PMATH 352 Notes

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1 January 6th

We study differential and integral calculus of complex functions of complex variables.
Replace $\mathbb{R}$ with $\mathbb{C}$.

- $\mathbb{C} \cong \mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$
  $x + iy \iff (x, y)$

This necessarily involves calculus of functions of two real variables.
The results will be much richer and deeper when we define complex differentiability.
We will see that notions like **regularity**, and **analyticity** are very different this time.
We will see a "unification" of close relationship of exponential and trigonometric, hyperbolic functions.
We will see that questions of real variable calculus can be answered by passing through the complex domain.
We will see close relationship between complex differentiable functions and harmonic functions in two variables. (We will start here)
Prereq: Previous exposure to real analysis also multivariable calculus.
It includes:

- Double integrals,
- Partial derivative,
- Directional derivative,
- Gradient,
- Chain rule.

1.1 Topology of $\mathbb{R}^2$

$$\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}^2\}$$

is two-dimentional real vector space.
Dot product also called Euclidean inner product.

$$z_1 = (x_1, y_1)$$
$$z_2 = (x_2, y_2)$$

$$z_1 \cdot z_2 = x_1x_2 + y_1y_2$$

$$z = (x, y)$$
$$|z|^2 = z \cdot z = x^2 + y^2 \geq 0$$

with equality when $z = (0, 0)$

$$|z| = \sqrt{(x^2 + y^2)}$$

Distance from 0 to z.
Cauchy-Schwarz Inequality

$$|z \cdot w| \leq |z| \cdot |w|$$

with equality when z, w are linearly dependent.
Pictures here.
Distance from z_0 to z is $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$
Triangle Inequality

$$|z - w| \leq |z - u| + |u - w|$$

Pictures here.
Define: Let $z_0 = (x_0, y_0) \in \mathbb{R}^2$. Let $\epsilon > 0$.

$$D(z_0; \epsilon) = \{z \in \mathbb{R}^2 : |z - z_0| < \epsilon\}$$

called the open disc of radius ϵ centered at z_0.
Define:

$$\overline{D(z_0; \epsilon)} = \{z \in \mathbb{R}^2 : |z - z_0| \leq \epsilon\}$$

closed disc of radius ϵ centered at z_0
Define:
Let $\Omega \subseteq \mathbb{R}^2$ be a subset.
Let $z \in \Omega$.
We say z is an interior point of Ω if $\exists \epsilon > 0$ such that $D(z; \epsilon) \subseteq \Omega$
Define:
$\Omega \subseteq \mathbb{R}^2$ is an open set if every point in Ω is an interior point of Ω.
Remarks:

1. $\emptyset$ is open.
2. $\mathbb{R}^2$ is open.
3. An open disc is an open set.
4. A closed disc is not an open set.

Example:

$$\Omega = D(z_0; \epsilon) \setminus \{z_0\}$$

is called a punctured disc.
Facts: (Exercise)

1. If Ω_1, Ω_2 are open $\Rightarrow \Omega_1 \cap \Omega_2$ is open.

   Hence any finite intersection of open sets is open.

2. If Ω_α is open $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha$ is open. (A can be uncountable.)

Connectedness
Definition from Topology by Munkres:
Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.
Another way of formulating the definition of connectedness is the following:
A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.
Definition:
Let $E \subseteq \mathbb{R}^2$.
We say that E is disconnected if $\exists$ open sets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^2$ such that $E \cap \Omega_1 \neq \emptyset, E \cap \Omega_2 \neq \emptyset$.
$E \cap \Omega_1 \cap \Omega_2 = \emptyset$
$E = (E \cap \Omega_1) \cup (E \cap \Omega_2)$
Informally, E is disconnected if it is "made up of more than one piece".
We say E is connected if it is not disconnected.
Fundamental Fact:
Let $f : E \to \mathbb{R}$ be continuous on E.
If E is connected, then $f(E) = \{f(z) : z \in E\}$ is an interval.
Corrolary: Intermediate Value Theorem.
Suppose $f : E \to R$ is continuous on E and E is connected.
Let $z_1, z_2 \in E$.
Let $f(z_1) = t_1$, $f(z_2) = t_2$.
$\forall t$ between t_1, t_2 $\exists z \in E$ such that $f(z) = t$.
Define:
A domain Ω in $\mathbb{R}^2$ is an non-empty open connected set.
Theorem:
Let $\Omega \subseteq \mathbb{R}^2$ be **open** (non-empty) then TFAE

1. Ω is connected
2. Any pair of points in Ω can be linked by a path made of a finite number of straight line segments each lying entirely in Ω

Equivalence!!!
Proof:
Let $z_0 \in \Omega$.
We want to show $S = \Omega$.
Claim: S is open.
Let $z \in S$. Since $S \subseteq \Omega, z \in \Omega$.
Ω is open, so $\exists \epsilon > 0$ such that $D(z, \epsilon) \subseteq \Omega$.
Hence for any $w \in D(z : \epsilon)$, $\exists$ a straight line from z to w.
So $\exists$ a piecewise linear path from z_0 to w.
So $w \in S$.
Hence $D(z, \epsilon) \subseteq S$.
S is open.
Claim:
$\Omega \setminus S$ is open suppose $w \in \Omega \setminus S$.
Since $w \in \Omega$, and Ω is open, $\exists \epsilon > 0$ such that $D(w, \epsilon) \subseteq \Omega$.
If $D(w, \epsilon) \cap S \neq \emptyset$, then $\exists$ piecewise linear path from z_0 to a point in $D(w, \epsilon)$ and hence to w, contradicting $w \notin S$.
Hence, $D(w, \epsilon) \cap S = \emptyset$.
So $D(w, \epsilon) \subseteq \Omega \setminus S$.
Hence $\Omega \setminus S$ is open.
$E : \Omega$.
$\Omega_1 = S$ open.
$\Omega_2 = \Omega \setminus S$ is open.
$\Omega_1 \cap E = S \neq \emptyset$.
$\Omega_2 \cap E = \Omega \setminus S$.
$E = \Omega = \Omega_1 \cup \Omega_2$.
But Ω is connected.
So at least one of Ω_1, Ω_2 empty.
But $S \neq \emptyset$. So $\Omega \setminus S = \emptyset \Rightarrow \Omega = S$.
This proves (1) $\Rightarrow$ (2).
(2) $\Rightarrow$ (1) (Sketch)
Suppose (2) holds but Ω is not connected.
Let Ω_1, Ω_2 be a disconnection of Ω.
$\Omega_1 \cap \Omega_2 = \emptyset$
$\Omega = \Omega_1 \cup \Omega_2$
$\Omega_1, \Omega_2 \neq \emptyset$.
$\exists z \in \Omega_1, w \in \Omega_2$.
$2 \Rightarrow \exists$ piecewise linear path from z to w. $\alpha : [0, 1] \to \mathbb{R}^2$ continuous, $\alpha(0) = z, \alpha(1) = w$.
$\Rightarrow \alpha([0, 1])$ is connected.

2 January 8th

Continue from last time.
Assume 2 holds, but Ω is not connected. Then, $\exists$ open subsets Ω_1, Ω_2 of $\mathbb{R}^2$ such that $\Omega_1 \neq \emptyset, \Omega_2 \neq \emptyset, \Omega_1 \cap \Omega_2 = \emptyset, \Omega = \Omega_1 \cup \Omega_2$
Picture here.
$\exists$ a piecewise linear path from z_1 to z_2 lying entirely in Ω.
Hence, $\exists$ a continuous map, $\alpha : [0,1] \to \mathbb{R}^2$ such that $\alpha(t) \in \Omega \forall t \in [0,1]$ and $\alpha(0) = z_1, \alpha(1) = z_2$.
$E = \alpha([0,1])$ is connected subset of $\mathbb{R}^2$.

$$E \cap \Omega_1 \neq \emptyset, E \cap \Omega_2 \neq \emptyset, E \cap \Omega_1 \cap \Omega_2 = \emptyset$$

Hence, Ω_1, Ω_2 give a disconnection of E.
Contradiction
So $\Omega$is connected.
Recall
A domain is a connected open subsets of $\mathbb{R}^2$.
Corollary:
We have lots of domains.
Examples:
Any open convex set is a domain.
Hence, an open disc $D(z;\epsilon)$ is a **domain**.
A punctured open disc $D(z;\epsilon) \setminus \{z\}$ is a domain.
An **annulus**, $\{z \in \mathbb{R}^2, R_1 < |z - z_0| < R_2\}$ for $R_2 > R_1 > 0$ is a domain.
Boundedness
Definition:
A subset E of $\mathbb{R}^2$ (need not be open) is called **bounded** if $\exists$ $R > 0$ such that

$$E \subseteq \overline{D(0;R)}$$

(The location of the disc is irrelevant) Example:

$$|z| \leq R, \forall z \in E$$

(A bounded set doesn't "go off to infinity")
Definition:
$E \subseteq \mathbb{R}^2$ is called **closed** if $\mathbb{R}^2 \setminus E$ is **open**.
$\Rightarrow$ arbitrary intersections of closed sets are closed.
$\Rightarrow$ finite unions of closed sets are closed.
In general a subset need not be open nor closed.
Definition:
$E \subseteq \mathbb{R}^2$ is **compact** if it is **closed** and **bounded**.
(Heine-Borel Theorem)
Fundamental Fact:
The continuous image of a compact set is compact.
Corollary:

Extreme Value Theorem
Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuous on E and suppose E is conpact. Then f attains a global max and global min on E
Example:

$$\exists z_1, z_2 \in E \text{ such that } f(z_1) \leq f(z) \leq f(z_2) \forall z \in E$$

Boundary of a set
Definition:
Let E be a subset of $\mathbb{R}^2$.
A point $z \in \mathbb{R}^2$ is called a **boundary point** of E iff

$$\forall \epsilon > 0, \text{ both } D(z; \epsilon) \cap E \neq \emptyset \text{ and } D(z; \epsilon) \cap (\mathbb{R}^2 \setminus E) \neq \emptyset$$

(i.e z is a boundary point of E iff **any** open neighbourhood of z contains both points in E and points not in E)
Example:
$E = D(w; r)$
The boundary points of E are the points $z \in \mathbb{R}^2$ such that $|z - w| = r$
None of the boundary points are in the set.
$\overline{D(W; r)} = \{z \in \mathbb{R}^2, |z - w| \leq r\}$
The boundary points of this set are the same as the previous example.
All the boundary points are in the set.
Picture here.
Some of the boundary points are in the set.
Clear:
A subset E is

1. Open iff it contains non of its boundary points
2. Closed iff it contains all of its boundary points

Definition:

$$\partial E = \{z \in \mathbb{R}^2 : z \text{ is a boundary point of } E\}$$

is called the (topological) boundary of E.
Notice:

$$\partial E = \partial \left(\mathbb{R}^2 \setminus E\right)$$

Boundary has nothing to do with boundedness.
Example:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

"The upper half plane"

$$\partial \Omega = \{(x, 0) : x \in \mathbb{R}\}$$

x - axis.

But is not bounded.

Curves in $\mathbb{R}^2$

Definition:

A **smooth curve** in $\mathbb{R}^2$ is a map

$$\alpha : [a, b] \to \mathbb{R}^2$$

such that

$$\alpha(t) = (\alpha_1(t), \alpha_2(t))$$

1. α is C^1 or $[a, b]$ (continuously differentiable)

   i.e

   $$\alpha_1 : [a, b] \to \mathbb{R}, \alpha_2 : [a, b] \to \mathbb{R}$$

   are continuous differentiable and their derivatives α_1', α_2' are continuous on $[a, b]$

   (Use one-sided limits at end points)

2. $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t)) \neq (0, 0) \forall t \in [a, b]$

$\alpha'(t)$ is called the velocity vector of the curve α at $\alpha(t)$

Example:

$$\alpha(t) = (R\cos(t), R\sin(t)), t \in [0, 2\pi], R > 0$$
$$\alpha'(t) = (-R\sin(t), R\cos(t))$$
$$|\alpha'(t)| = R > 0$$

$$\begin{aligned}\gamma(t) &= (x_0 + R\cos t, y_0 + R\sin t) \\ &= \text{ circle of radius } R \text{ centered at } (x_0, y_0)\end{aligned}$$

Example:

$\alpha(t) = (R\cos(2\pi - t), R\sin(2\pi - t)), 0 \leq t \leq 2\pi$

"Opposite direction"

Example:

$\alpha(t) = (R\cos t, R\sin t), t \in [0, 4\pi]$

"Same" as example 1, but travels aroudn the image twice.

So it is a different curve.

Example:

$\alpha(t) = (R\cos(2t), R\sin(2t)), 0 \leq t \leq \pi$

Same image as all the others, same "orientation" as the examples 1, 3. Only goes around once like example 1.

But it goes around twice as fast.

So, it is a different curve.

We need a slight generalization!

Definition:

A **piecewise-smooth curve** in $\mathbb{R}^2$ is a continuous map $\alpha : [a, b] \to \mathbb{R}^2$ and a decomposition $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ such that $\alpha_i := \alpha|_{[t_i, t_{i+1}]}$, $[t_i, t_{i+1}] \to \mathbb{R}^2$ is a smooth curve $\forall i = 0, 1, \ldots, N > 1$
Note:
Continuity says

$$\alpha_i(t_{i+1}) = \alpha_{i+1}(t_{i+1})$$

From now on, a curve means a piecewise-smooth curve.
Definition:
A curve, $\alpha : [a, b] \to \mathbb{R}^2$ is called **simple** is $\alpha|_{(a,b)}$ is one-to-one.
i.e $\alpha(t_1) \neq \alpha(t_2)$ except **possibly** for $t_1 = a, t_2 = b$
(Example 3 where we traverse the circle twice is not simple.)
Definition:
$\alpha : [a, b] \to \mathbb{R}^2$ is called **closed** if $\alpha(a) = \alpha(b)$.
(final point equals initial point)
Examples:
Pictures here.

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Length of a (piecewise smooth) curve
Let α be a curve,

$$\alpha[a, b] \to \mathbb{R}^2$$

Define:
The **length** of α is

$$L(\alpha) = \int_a^b |\alpha'(t)|\, dt$$

α is continuous. $L(\alpha) > 0$ since $|\alpha'(t)| > 0, \forall t \in [a, b]$
Example:

$$\alpha(t) = (R\cos t, R\sin t), 0 \leq t \leq 2\pi$$

$$\alpha'(t) = (-R\sin t, R\cos t)$$

$$|\alpha'(t)| = R$$

$$L(\alpha) = \int_0^{2\pi} R\, dt = 2\pi R$$

Definhition:
A **reparametrization** of a curve $\alpha : [a, b] \to \mathbb{R}^2$ is a bijective continuous map $h : [c, d] \to [a, b]$ such that

1. h is piecewise-smooth

2. either $\begin{cases} t = h'(s) > 0 \ \forall s \in [c,d] & \text{Orientation (direction) preserving} \\ t = h'(s) < 0 \ \forall s \in [c,d] & \text{Orientation reversing} \end{cases}$

So that $\tilde{\alpha} : [c,d] \to \mathbb{R}^2$ by

$$\tilde{\alpha}(s) = \alpha(h(s)), \forall s \in [c,d]$$

then
$\tilde{\alpha} : [c,d] \to \mathbb{R}^2$ is a piecewise-smooth curve whose image is the same as α and that passes through each point in the image the same number of times as α and in

1. (2a) Same direction
2. (2b) Opposite direction

Chain Rule:

$$\tilde{\alpha}'(s) = \alpha'(h(s))h'(s)$$

so $\tilde{\alpha}$ is picewise smooth.

1. $h(c) = a, h(d) = b$
2. $h(c) = b, h(d) = a$

Picture here.
Proposition:
Let $\tilde{\alpha}(s) = \alpha(h(s))$ be a reparametrization of α, then $L(\tilde{\alpha}) = L(\alpha)$
$t = h(s)$
Proof:

$$\tilde{\alpha}'(s) = \alpha'(h(s))h'(s)$$
$$|\tilde{\alpha}'(s)| = |\alpha'(h(s))||h'(s)|$$

$$
\begin{aligned}
L(\tilde{\alpha}) &= \int_c^d |\tilde{\alpha}'(s)|\ ds \\
&= \int_c^d |\alpha'(h(s))||h'(s)|\ ds \\
(2a) &= \int_c^d |\alpha'(h(s))|\frac{dt}{ds}\ ds \\
&= \int_a^b |\alpha'(t)|\ dt \\
&= L(\alpha) \\
&= (2b) - \int_c^d |\alpha'(h(s))|\frac{dt}{ds}\ ds \\
&= -\int_a^b |\alpha'(t)|\ dt \\
&= L(\alpha)
\end{aligned}
$$

Theorem:
Reparametrization by arclength
Let $\alpha : [a,b] \to \mathbb{R}^2$ be a curve. Let $L = L(\alpha) > 0$
$\exists 1$ orientation preserving reparametrization

$$h : [0, L] \to [a,b](h'(s) > 0, \forall s \in [0, L])$$

such that $\tilde{\alpha} = \alpha \cdot h$ has unit speed

$$|\tilde{\alpha}'(s)| = 1\ \forall s \in [0, L]$$

Suppose we had this

$$s_0 \in [0, L]$$

$$\int_0^{s_0} |\tilde{\alpha}'(s)|\ ds = \text{ length of } \tilde{\alpha}|_{[0,s0]} = \int_0^{s_0} 1\ ds = s_0$$

So $s_0 \in [0, L]$ is the length of $\tilde{\alpha}(0) = \alpha(a)$ to $\tilde{\alpha}(s_0) = \alpha(h(s_0))$.
Proof:
First, suppose α is smooth, let $L = L(\alpha) = \int_a^b |\alpha'(t)|\ dt$
We seek a function

$$h : [0, L] \to [0, b]$$

bijection such that

$$\tilde{\alpha}(s) = \alpha(h(s))$$

has unit speed.

$$t = h(s)$$
$$s = h^{-1}(t) = f(t)$$

Length of the curve from $\alpha(0) = \tilde{\alpha(0)}$ to $\tilde{\alpha}(s) = \tilde{\alpha} f(t) = \alpha(t)$
$f = h^{-1} : [a, b] \to [0, L]$
Hence
$$f(t) = \int_a^t |\alpha'(u)|\, du$$
$$f(a) = 0$$
$$f(b) = L = \text{Length}(\alpha)$$

Note: $|\alpha'(u)|$ is continuous.
So by Fundamental Theorem of Calculus, $f(t)$ is differentiable.
$$f'(t) = |\alpha'(t)| > 0, \forall t \in [a, b]$$

So by Calculus I,
There exists an inverse function $h = f^{-1}$ such that $f(h(s)) = s, \forall s$ (Differentiable with respect to s).
$$f'(h(s))h'(s) = 1$$
$$\begin{aligned} h'(s) &= \frac{1}{f'(h(s))} \\ &= \frac{1}{f'(t)} = \frac{1}{|\alpha'(t)|} \end{aligned}$$
$$h(f(t)) = t, \forall t$$

and h is C^1 on $[a, b]$.
Set $\tilde{\alpha}(s) = \alpha(h(s))$.
$$\begin{aligned} \tilde{\alpha}'(s) &= \alpha'(h(s))h'(s) \\ &= \alpha'(t)\frac{1}{|\alpha'(t)|} \end{aligned}$$

Hence $|\tilde{\alpha}'(s)| = 1, \forall s$
We have proved it for smooth curves.
Suppose α is piecewise smooth
$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_N = b$$
$$\alpha_i = \alpha|_{[t_i, t_{i+1}]}$$
is smooth $i = 0, \ldots, N-1$

Let h_i be the "h" for α_i define h by

$$\begin{aligned} h(s) &= h_0(s), 0 \le s \le L(\alpha_0) \\ &= L(\alpha_0) + h_1(s), L(\alpha_0) \le s \le L(\alpha_0) + L(\alpha_1) \end{aligned}$$

etc.
Exercise:

$$\alpha(t) = (R\cos t, R\sin t)\,, t \in [0, 2\pi]$$

We want to reparametrize by arclength

$$s = f(t) = \int_0^t |\alpha'(u)|\, du = Rt$$

$$t = \frac{s}{R}$$

$$\tilde{\alpha}(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right)$$

$$0 \le s \le 2\pi R$$

From now on we can WLOG assume that any curve is paramerized by arclength.
Define:
A **Jordan curve** is a curve $\alpha : [a, b] \to \mathbb{R}^2$ that is simple and closed. $t_1 < t_2, \alpha(t_1) \neq \alpha(t_2), \alpha(a) = \alpha(b)$
Theorem:
Jordan curve Theroem
Let $\alpha : [a, b] \to \mathbb{R}^2$ be a **Jordan curve** with image $\Gamma = \alpha\left([a, b]\right)$.
Then $\mathbb{R}^2 \setminus \Gamma$ consists of two **disjoint domains**.
One of which is bounded (called "inside") and the other is unbounded (called "outside"). Each domain has Γ as its boundary.
If a point inside Γ is joined to a point outside Γ by a curve, then that curve must intersect Γ.
We won't prove this. It is intuitively clear but requires some algebraic topology to prove. (PMATH 365)
Definition:
Jordan Domain
A Jordan domain is a bounded domain (open + connected) Ω such that its boundary is the union of finitely many images of Jordan curves.
We choose to orient each of these Jordan curves so that as we traverse the curve in the direction of its orientation, the Jordan domain Ω lies on left side.
Picture here.
0-connected
1-connected
3-connected

4 January 13th

Outward normal vector field N to a curve
Let Ω be a Jordan domain with boundary $\partial\Omega$.
Let Γ be one of the Jordan curves in $\partial\Omega$, without loss of generality, let Γ be paramerized by arclength.

$$\alpha(s) = (\alpha_1(s), \alpha_2(s)), s \in [0, L], L = \text{Length}(\alpha)$$

$$\begin{aligned} T(s) &= \alpha'(s) \\ &= (\alpha_1'(s), \alpha_2'(s)) \\ &= \text{unit tangent vector field to } \alpha \end{aligned}$$

$$|T(s)| = |\alpha'(s)| = 1$$

Definition:
The outward unit normal vector field N to α is defined to be

$$N(s) = (\alpha_2'(s), -\alpha_1'(s))$$

$$|N(s)| = 1, \forall s \in [0, L]$$

By our choice, N is pointing outward at all points on Γ.
Examples:
$\alpha(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right), 0 \le s \le 2\pi R$
$T(s) = \alpha'(s) = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$
$N(s) = \left(\cos\left(\frac{s}{R}\right), \sin\frac{s}{R}\right)$
Picture example.
Read 1.3 on your own.
The outward normal derivative
Let Ω be the Jordan domain, let $z_0 \in \partial\Omega$.
Let $N(z_0)$ be the outward pointing unit normal vector.
Picture here.
Let W be an open set in $\mathbb{R}^2$. Such that

$$W \subseteq \Omega \cup \partial\Omega = \overline{\Omega}$$

Let $u \in C^1(W), u \cdot W \subseteq \mathbb{R}^2 \to \mathbb{R}$, continuously differentiable, $u_x, u_y \in C^0(W)$.
Definition:

$$\frac{\partial u}{\partial n}(z_0) := D_{N(z_0)}u = (\nabla u)(z_0) \cdot N(z_0)$$

$D_{N(z_0)}u$: Directional derivative of u at z_0 in $N(z_0)$ direction.
Example: Assignment 1
If $\alpha(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\frac{s}{R}\right)$

Then:

$$\frac{\partial u}{\partial n}(z_0) = \frac{\partial u}{\partial r}(z_0)$$

Picture here.
This is the partial derivative of u at z_0 with the polar coordinates r.
The Laplacian Let $W \subseteq \mathbb{R}^2$ be open.
Let $u \in C^2(W)$.

$$u_{xx}, u_{xy} = u_{yx}, u_{yy} \in C^0(W)$$

Then we define the **Laplacian of** u, denoted Δu, by

$$\Delta u = u_{xx} + u_{yy} \in C^0(W)$$

Define: $u \in C^2(W)$ is called **harmonic** on W if $\Delta u = 0$ on W.
In Chapter 2, we will see harmonic functions have very nice properties and closely related to complex analysis.
1.4.1 Line integrals of vector fields
Define:
Let $\Omega \subseteq \mathbb{R}^2$ be open.
A **vector field** F on $Omega$ is a map

$$F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

$$F(x, y) = (P(x, y), Q(x, y))$$

$$P, Q : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$$

We say F is a C^k vector field on Ω iff both $P, Q \in C^k(\Omega)$.
We always assume F is at least C^0 vector field.
Exercise 1

$$F(x, y) = (-y, x)$$

$\Omega = \mathbb{R}^2$
Exercise 2

$$G(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right), \Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$$

Let $\alpha : [0, h] \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$ suppose $Im(\alpha) \subseteq \Omega$.
Let F be a C^0 vector field on Ω.

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (x(t), y(t)), a \leq t \leq b$$

Let $\alpha : [a, b] \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$. Suppose $\mathrm{Im}(\alpha) \subseteq \Omega$.
Let F be a C^0 vector field on Ω.
Definition:

The **line integral** of the vector field F along the curve α is defined to be

$$\int_{\alpha} F{\cdot}dr := \int_a^b F(\alpha(t)){\cdot}\alpha'(t)\ dt = \int_a^b P(x(t),y(t))x'(t)\ dt{+}\int_a^b Q(x(t),y(t))y'(t)\ dt$$

Notation:
The authour writes

$$\int_{\alpha}(Pdx + Qdy)$$

Examples:

$$\alpha(t) = (R\cos(t), R\sin(t)), 0 \leq t \leq 2\pi$$

$$\int_{\alpha} F \cdot dr$$

See pictures.

$$F(\alpha(t)) = (-R\sin t, R\cos t)$$
$$\alpha'(t) = (-R\sin t, R\cos t)$$
$$F(\alpha(t)) \cdot \alpha'(t) = R^2$$
$$\int_{\alpha} F \cdot dr = 2\pi R^2$$
$$\int_{\alpha} G \cdot dr$$

$$G(\alpha(t)) = (\cos(t), \sin(t))$$
$$\alpha'(t) = (-R\sin t, R\cos t)$$
$$G(\alpha(t)) \cdot \alpha'(t) = 0$$
$$\int_{\alpha} G \cdot dr = 0$$

Proposition:
$\int_{\alpha} F \cdot dr$ is independent of reparametrization of α as long as the orientation is presented.
Proof:
Let $\tilde{\alpha}(s) = \alpha(h(s)), 0 \leq s \leq d$ be a reparametrization.

$$\tilde{\alpha}'(s) = \alpha'(h(s))h'(s)$$

Chain rule

$$\int_{\alpha} F \cdot dr = \int_a^b F(\alpha(t)) \cdot \alpha'(t)\ dt$$

Let $t = h(s)$

$$t = a \iff s = c$$
$$t = b \iff s = d$$

(Orientation preservirs)

$$dt = h'(s)ds$$

$$\begin{aligned} &= \int_c^d F\left(\alpha(h(s))\right) \cdot \alpha'(h(s))h'(s)\ ds \\ &= \int_c^d F\left(\tilde{\alpha}(s)\right) \cdot \tilde{\alpha}'(s)\ ds \\ &= \int_{\tilde{\alpha}} F \cdot dr \end{aligned}$$

From the proof, it is clear that

$$\int_{-\alpha} F \cdot dr = -\int_{\alpha} F \cdot dr$$

So because of this proposition, WLOG we can assume (if necessary) that α is parametrized by arclength (only its orientation matters).
What is the geometric / physical meaning of $\int_\alpha F \cdot dr$? Assume α is unit speed.

$$\int_\alpha F \cdot dr = \int_0^b F(\alpha(s)) \cdot \alpha'(s)\ ds$$

Component of F along the unit tangent vector field.

$$T(s) = \text{ unit vector field along } \alpha$$

This clarifies the two examples.

5 January 15th

Green's Theorem and Green's Identities
This is the basic tool that will imply most of the big results in this course.
Theorem:
Let Ω be a k-connected Jordan domain and let $F(x, y) = (P(x, y), Q(x, y))$ be a C^1 vector field on a domain Ω^+ which contains Ω and $\partial\Omega$.
Then

$$\int_{\partial\Omega} F \cdot dr = \iint_\Omega \left(Q_x - P_y\right)\ dA$$

Picture here.
Proof:

We will prove it for a rectangle and then explain how the general case follows from that.

$$F(x,y) = (P(x,y), Q(x,y))$$

On curve (1),

$$\alpha(t) = (t, c), a \le t \le b$$
$$\alpha'(t) = (1, 0)$$
$$F(\alpha(t)) = (P(t,c), Q(t,c))$$

$$\int_{(1)} F \cdot dr = \int_a^b P(t,c)\ dt$$
$$\int_{(3)} F \cdot dr = -\int_a^b P(t,d)\ dt$$

Or (2)

$$\alpha(t) = (b, t), c \le t \le d$$
$$\alpha'(t) = (0, 1)$$
$$F(\alpha(t)) = (P(b,t), Q(b,t))$$

$$\int_{(2)} F \cdot dr = \int_c^d Q(b,t)\ dt$$
$$\int_{(4)} F \cdot dr = -\int_c^d Q(a,t)\ dt$$

$$\int_{\partial\Omega} F \cdot dr = \int_a^b P(t,c)\ dt - \int_a^b P(t,d)\ dt + \int_c^d Q(b,t)\ dt - \int_c^d Q(a,t)\ dt$$

$$\begin{aligned} \iint_{\Omega} (Q_x - P_y)\ dA &= \iint_{\Omega} Q_x\ dA - \iint_{\Omega} P_y\ dA \\ &= \int_c^d \int_a^b \frac{\partial Q}{\partial x}\ dx\ dy - ... \end{aligned}$$

Green's Theorem is true for rectangles.
General Case:
Pictures here.
Green's Identities
Consequence of Green's Theorem
We first need some notation.
For the rest of the lecture, Ω is a k-connected Jordan domain, and Ω^+ is a domain containing $\Omega \cup \partial\Omega = \overline{\Omega}, u, v \in C^2(\Omega^+), u, v : \Omega^+ \to \mathbb{R}$.

Definition:
We want to define $\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds$
It is also the line integral along $\partial\Omega$ of the vector field $(-u_y, u_x)$ which $\nabla u = (u_x, u_y)$ rotated 90 degrees counterclockwise
$\frac{\partial u}{\partial n}$: Outward notmal derivative of u on $\partial\Omega$.
as follows

$$F(x,y) = (-u_y, u_x)$$

is a C^1-vector field.
Let $\alpha : [0,1] \to \mathbb{R}^2$ be the arclength parametriaztion of $\partial\Omega$.

$$\begin{aligned}
\int_\alpha F \cdot dr = \int_0^L F(\alpha(t)) \cdot \alpha'(t)\, dt &= \int_0^L (-u_y x' + u_x y')\, dt \\
&= \int_0^L (u_x, u_y)(y', -x)\ dt \\
&= \int_0^L (\nabla u)\, \alpha(t) \cdot N(t)\, dt \\
&= \int_0^L \frac{\partial u}{\partial n}(t)\, dt
\end{aligned}$$

If $f \in C^2(\Omega^+)$, then

$$\int_{\partial\Omega} f \frac{\partial u}{\partial n}\, ds = \text{line integrals of } (-fu_y, fu_x)$$

Green's Identity #1

$$\iint_\Omega (\nabla u) \cdot (\nabla v)\ dA = \int_{\partial\Omega} u \frac{\partial v}{\partial n} ds - \iint_\Omega u \Delta v\, dA$$

Proof:
We will explicitly evaluate the $\int_{\partial\Omega}$ term.
... See picture.
Green's Second Identity

$$\int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} ds = \iint_\Omega (v\Delta u - u\Delta b)\, dA \right)$$

Proof:
Interchanging u and v in 1st identity and take difference.
Corollary: Inside-Outside Theroem

$$\int_{\partial\Omega} \frac{\partial v}{\partial n}\, ds = \iint_\Omega \Delta v\, dA$$

Lemma:
Let $z_0 = (x_0, y_0) \in \mathbb{R}^2$ be fixed.
Let $z = (x, y)$ be variable point.

Define $r(z) = r(x,y) = |z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$
$\log r$ is **harmonic** on $\mathbb{R} \setminus \{z_0\}$.
Proof:
Calculate $(\log r)_{xx} + (\log r)_{yy} = 0$
See picture.
Green's Third Identity
Fix $z_0 \in \Omega, r(z) = |z - z_0|$.

$$u(z_0) = \frac{1}{2\pi} \iint_\Omega \log r \delta u \; dA - \frac{1}{2\pi} \int_{\partial\Omega} \left(\log r \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} (\log r) \right) \; ds$$

Remark:
This is remarkable. It says that given knowledge of u and $\frac{\partial u}{\partial n}$ on $\partial\Omega$ and δu in Ω, it determines u inside Ω.
Proof:
Let $\epsilon > 0$, such that $\overline{D(z_0), \epsilon} \subseteq \Omega$.
We can do this by taking ϵ sufficiently small.
Apply Green's Second Identity to $\Omega \setminus \overline{D(z_0, \epsilon)} \ldots$
See pictures.

6 January 17th

From last time:
Green's Theorem.
Ω k-connected Jordan domain. Ω^+ domain such that $\Omega^+ \supseteq \Omega \cup \Omega$.
Let $u \in C^2(\Omega^+)$, $r = |z - z_0|$

$$u(z_0) = \frac{1}{2\pi} \iint_\Omega (\log r) \Delta u \; dA - \frac{1}{2\pi} \int_{\partial\pi} (\log r) \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} (\log r) \; ds$$

Inside-Outside Theorem

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \; ds = \iint_\Omega (\Delta u) \; dA$$

Recall:
If W is an open set, $u \in C^2(W)$, we say that u is **harmonic** on W if $\Delta u = u_{xx} + u_{yy} \in C^0(W) = 0$ on W.
Examples:

1. $u(x,y) = Ax + By + C$: Graph of u is a plane in $\mathbb{R}^3$
2. $u(x,y) = x^2 + y^2$, $u_{xx} = 2$, $u_{yy} = -2$, $\Delta u = 0$
3. $u(x,y) = xy, u_{xx} = u_{yy} = 0$
4. $u(x,y) = e^x \cos y, u_{xx} = e^x \cos y, u_{yy} = -e^x \cos y, \Delta u = 0$

We will soon see how the examples are related to complex differentiable functions.
Clear:
Any linear combination of harmonic functions on W is again harmonic on W because $\Delta : C^2(W) \to C^0(W)$ is a linear map.
Harmonic functions on $W = \mathrm{Ker}(\Delta)$
So harmonic functions on W are a real vector space (it is infinite-dimensional)
Theorem:
Suppose $*$ holds, let $u \in C^2\left(\Omega^+\right)$ be harmonic on Ω^+
Let $z_0 \in \Omega$, then

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = 0$$

$$u(z_0) = -\frac{1}{2\pi}\int_{\partial\pi} (\log r)\,\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}(\log r)\; ds$$

Remarks:
(A) says that for a harmonic function, the "net flux" across the entire boundary is zero.
(B) says that a harmonic function is determined by its **boudnary** behaviour.
(Knowledge of u, $\frac{du}{dn}$ on $\partial\Omega$ determines u inside $\partial\Omega$)
Read 2.2 for physical interpretation.
A characterization of Harmonicity
Theorem:
Let W be a domain, $u \in C^2(W)$.
Then u is harmonic on W if and only if for every Jordan curve, Γ, inside W, whose interior lies inside W, we have $\int_\Gamma \frac{\partial u}{\partial n}\, ds = 0$
Proof:
We already know 1) implies 2).
Picture here.
Conversely, suppose 2) holds. We need to show $\Delta u = 0$ on W. Suppose not $\exists z_0 \in W$ such that $\Delta u(z_0) \neq 0$ by possibly replacing u by $-u$, we can assume $\left(\Delta u\right)(z_0) > 0$.
But $\Delta u \in C^0(W)$, so $\exists \epsilon > 0$ such that $\left(\Delta u\right)(z) > 0, \forall z \in \overline{D(z_0,\epsilon)} \subseteq W$.
Then $\Gamma = \partial\Omega = C(z_0,\epsilon) = \{z : |z - z_0| = \epsilon\}$

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = \iint_\Omega (\Delta u)\; dA > 0$$

Contradiction.
Aside: Differentiation under integral sign
Let $R = \{a \leq s \leq b, c \leq t \leq d\}$ be a rectangle in $s - t$ plane.
Suppose $F(s,t), F_t(s,t)$ are continuous on an open set containing R.
Then,

$$\int_a^b F(s,t)\; ds$$

is a differentiable function of t on $a < t < b$ and

$$\frac{d}{dt}\int_a^b F(s,t)\, ds = \int_a^b F_t(s,t)\, ds$$

Proof:
Define $f(t) = \int_a^b F(s,t)\, ds, g(t) = \int_a^b F_t(s,t)\, ds$
We want to show that f is differentiable on (a,b) and $f'(t) = g(t)$

$$\begin{aligned}\int_c^\tau g(t)\, dt &= \int_c^\tau \int_a^b F_t(s,t)\, ds\, dt \\ &= \int_a^b \int_c^\tau F_t(s,t)\, dt\, ds \\ &= \int_a^b F(s,\tau) - F(s,c)\, ds \\ &= f(t) + \text{constant}\end{aligned}$$

Take $\frac{d}{dt}$ use FTC.

$$g'(\tau) = f(\tau) \forall \tau \in (a,b)$$

Corollary:
Let Ω be a domain in $\mathbb{R}^2$, $u \in C^2(\Omega)$ harmonic on Ω.
Then in fact $u \in C^\infty(\Omega)$
Proof:
(Theorem B):
If $(x,y) \in \Omega$

$$u(x,y) = -\frac{1}{2\pi}\int_{\partial D}\left(\log r \frac{\partial u}{\partial n}(\alpha(s)) - u(\alpha(s))\frac{\partial}{\partial n}(\log r)\right)\, ds$$

When D is a disc centered at z_0, with $\overline{D} = D \cup \partial D \subseteq \Omega$, and $\alpha : [0,1] \to \mathbb{R}^2$ is an arclength parametriaztion for ∂D.

$$\alpha(s) = (x(s), y(s))$$
$$\alpha'(s) = (x'(s), y'(s))$$

$$r(x,y,s) = |(x,y) - (\alpha_1(s), \alpha_2(s))|$$

$$(x,y) \notin \partial D$$

Keep differentiating as many times as your want. Using differentiation under integral.
Mean Value Property of Harmonic Functions
Theorem: Circumferential Mean Value Theorem
Let u be harmonic in a domain Ω.

Suppose $\overline{D(z_0, R)} \subseteq \Omega$.
Then

$$u(z_0) = \frac{1}{2\pi R} \int_{C(z_0,R)} u(z)\ ds = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta)\ d\theta$$

Polar coordinates centered at z_0.
To see that these two are equal on $C(z_0, \epsilon)$, arclength $s = R\theta, ds = Rd\theta, 0 \leq \theta \leq 2\pi$.
The left hand side says that the value of u at z_0 equals the average of the values of u on the circumference $C(z_0; R)$ of $D(z_0; R)$
Proof:

1. We first show that the integral

$$\frac{1}{2\pi R} \int_{C(z_0;R)} u(z)\ ds$$

is independent of R as long as $\overline{D(z_0, R)} \subseteq \Omega$.

Let $0 < r \leq R$.

Let $C(z_0; r)$ be circle of radius r centered at z_0 (counterclockwise).

Since u is harmonic on Ω, h_y the characterization of harmonity,

$$0 = \int_{C(z_0,r)} \frac{\partial u}{\partial n}\ ds = \int_{C(z_0,r)} \frac{\partial u}{\partial r}\ ds = \int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta) r\ d\theta$$

$$= r \int_0^{2\pi} \frac{\partial u}{\partial r}(r, \theta)\ d\theta$$

$$= r \frac{d}{dr} [\int_0^{2\pi} u(r, \theta)\ d\theta]$$

constant in r for $0 < r \leq R$

$$= 2\pi u(r, \theta), 0 \leq \theta_r \leq 2\pi$$

Take limit as $r \to 0$.

$$= \int_0^{2\pi} u(R, \theta)\ d\theta$$

$$\lim_{r \to 0} \int_0^{2\pi} u(r, \theta)\ d\theta$$

Limit as $r \to 0^+$

$$= 2\pi u(r, \theta_r)$$

$$= 2\pi u(z_0)$$

by the continuity of u.

7 January 20th

From last time:
Let Ω be a Jordan domain.
Let Ω^+ be domain, $\Omega^+ \subseteq \Omega \cup (\partial\Omega)$.

- Suppose $u \in C^2(\Omega^+)$ is harmonic on Ω^+

  Let $z_0 \in \Omega$

  $$u(z_0) = -\frac{1}{2\pi}\left(\int_{\partial\Omega} \log r \frac{\partial u}{\partial n} - i\frac{\partial}{\partial n}\log r\right)\, ds$$

  where $r(z) = |z - z_0|$.

- u is harmonic of Ω^+ iff

  $$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = 0$$

  (Net flux)

  $\forall\Omega$ Jordan domain with $\Omega \cup \partial\Omega \subseteq \Omega^+$.

Theorem:
Circumferential Mean Value Property
Let u be harmonic in a domain Ω.
Suppose $\overline{D(z_0; R)} \subseteq \Omega$
Then,

$$u(z_0) = \frac{1}{2\pi R}\int_{C(z_0,R)} u(z)\, ds = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\, d\theta$$

.
Proof:
Second Proof:
Use result (1) with $\Omega = D(z_0, R)$ on the boundary of Ω, $\partial\Omega = C(z_0; R), r = R = |z - z_0| = \text{Const}$

$$u(z_0) = -\frac{1}{2\pi}\int_{C(z_0;R)} \log R\frac{\partial u}{\partial n} - u\frac{1}{R}\, ds$$

(Constant on $C(z_0, R)$)

$$= -\frac{1}{2\pi}\log R\int_{C(z_0,R)} \frac{\partial u}{\partial n}\, ds + \frac{1}{2\pi}\int_{C(z_0;R)} u\, ds$$

(The first term is 0 by 2)
Definition:
Let Ω be a domain.
Let $u \in C^2(\Omega)$. We say u satisfies the circumferential MVP in Ω iff $\forall\overline{D(z_0; R)} \subseteq \Omega$,

$$u(z_0) = \frac{1}{2\pi R}\int_{C(z_0;R)} u\, ds$$

So any harmonic function has the CMVP.
Theorem:
Let $u \in C^2(\Omega)$ for a domain Ω.
Then (1) u has the CMVP on Ω iff (2) u is harmonic on Ω
Proof:
We already know (1) $\Rightarrow$ (2).
Suppose (2) holds, let $\overline{D(z_0;R)} \subseteq \Omega$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta$$

Differentiate both sides with respect to r. (We know we can differentiate under integral)
And multiply by r.

$$0 = \frac{r}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial n}(r,\theta)\, d\theta$$

$$0 = \int_0^{2\pi} \frac{\partial u}{\partial n}(r,\theta) r\, d\theta = \int_{C(z_0,r)} \frac{\partial u}{\partial n}\, ds$$

Inside-outside theorem.

$$\int_{D(z_0,r)} \Delta u\, dA \quad \forall \overline{D(z_0;r)} \subseteq \Omega$$

$$\Rightarrow \Delta u = 0 \text{ on } \Omega$$

Theorem: Solid Mean Value Property
Let u be harmonic on a domain Ω. Let $\overline{D(z_0;R)} \subseteq \Omega$.
Then

$$u(z_0) = \frac{1}{\pi R^2}\iint_{D(z_0;R)} u\, dA$$

(Again, it says that the value of u at z_0 equals the average mean value of u over $\overline{D(z_0;R)} \subseteq \Omega$)
Proof:
Let $0 < r \leq R$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta$$

by CMVP.
Multiply both sides by r and integrate $0 \leq r \leq R$

$$\int_0^R u(z_0) r\, dr = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} u(r,\theta) r\, dr\, d\theta = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} u(r,\theta)\, dA$$

$$\frac{R^2}{2} u(z_0) = \frac{1}{2\pi}\iint_{D(z_0;R)} u\, dA$$

Theorem: (Strong Maximum Principle)
Let Ω be a domain in $\mathbb{R}^2$ [We do not assume that it is a Jordan domain nor even that is bounded]
Let u be harmonic on Ω and non-constant.
Then u does **not attain** a global max nor a global min on Ω.
Proof:
By contradiction. Suppose u has a global max at $z_0 \in \Omega$. That means $u(z) \leq u(z_0) = c\ \forall z \in \Omega$.
Let $U = \{w \in \Omega; u(w) < c\}$.
This is non-empty because u is non-constant.
and open since u is continuous.
Let $E = \{w \in \Omega, u(w) = c\}$. Non-empty $z_0 \in E$

$$\Omega = U \cup E$$

since $u(w) \leq c\ \forall w \in \Omega$
Hence, E cannot be open, because if it was, it would give a disconnection of Ω. (Ω is connected).
So E contains at least one of its boudnary points.
That is, there exists $z \in E$, $u(z_1) = c$ such that $\forall \epsilon > 0$, $D(z_1; \epsilon) \cap \left(\mathbb{R}^2 \setminus E\right) = \emptyset$
Take ϵ sufficiently emall so that $D(z_1, \epsilon) \subseteq \Omega$ (Since Ω is open).
There exists $\zeta \in D(z_1; \epsilon)$ such that $\zeta \in U, u(\zeta) < c$

$$|\zeta - z_1| = R < \epsilon$$

Since u is continuous, and $u(\zeta) < c$. There exists a whole ark of the circle $C(z_1; R)$ on which $u < c$.

$$u(R, \theta) < u(z_1)\ \forall \theta \text{ in some open arc and } u(R, \theta) \leq u(z_1)\ \forall \theta$$

Integrate in θ from 0 to 2π.

$$\int_0^{2\pi} u(R, \theta)\, d\theta < \int_0^{2\pi} u(z_1)\, d\theta = 2\pi u(z_1)$$

$$\frac{1}{2\pi} \int_0^{2\pi} u(R, \theta)\, d\theta < u(z_1)$$

contradicts CMVP.
Hence u does not attain a global max on Ω.
For the global min, the proof is similar, just change all the equalities. (Or apply the global max result to $-u$)
It is called the **STRONG** maximum principle because we do not need Ω to be bounded, nor do we need any hypothesis about u on $\partial\Omega$.
Theorem: Week Maximum Principle
Let Ω be a bounded domain. Let u be continuous $u \in C^0\left(\overline{\Omega}\right) \cap C^2\left(\Omega\right)$
and **harmonic** on Ω.

[$\overline{\Omega}$ is compact, $u \in C^0(\overline{\Omega})$ by EVT, u does havea global max and a global min on $\overline{\Omega}$]
Then: either u is constant, or u assumes its global max and min ONLY on the boundary.
Proof:
u is constant on $\Omega \Rightarrow u$ is constant on $\overline{\Omega}$ (by continuity)
Suppose u is not constant on Ω, by Strong Maximum Principle. It does not exist in global max/min on Ω.
Therefore, it is on the boundary? (Not sure what I heard)
Equivalently, Week Maximum Principle says:
If Ω is a bounded domain, $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is harmonic on Ω, then

$$\forall z \in \Omega \ \min_{w \in \partial\Omega} u(w) < u(z) < \max_{w \in \partial\Omega} u(w)$$

Example:
To see boundedness of Ω is essential:

$$\Omega = \{(x, y); y > 0\}$$

upper half plane.

$$\partial\Omega = \{(x, 0); x \in \mathbb{R}\}$$

x-axis.
$u(x, y) = y$ is harmonic, non-constant on Ω.
It has no global max on $\overline{\Omega}$.
Application:
Uniqueness of harmonic functions with given boudnary values.
Theorem:
Let Ω be a bounded domain, let $u, v \in C^0(\Omega) \cap C^2(\Omega)$ be harmonic on Ω with $u|_{\partial\Omega} = v|_{\partial\Omega}$.
Then $u = v$.
Proof:
Let $w = u - v$, $w \in C^2\left(\overline{\Omega}\right) \cap C^0(\Omega)$.
w is harmonic on Ω.
$w|_{\partial\Omega} = 0$
If w is non-constant, it contradicts WMP.
So w is constant on Ω, $w|_{\partial\Omega} = 0 \Rightarrow w = 0$ or Ω.

8 January 22nd

Recall: Weak Maximum Principle (Corollary of Strong Maximum Principle)
Ω **bounded** domain, $u \in C^0\left(\overline{\Omega}\right) \cap C^2(\Omega)$ is harmonic on Ω, **non-constant**.
Then,

$$\max_{z \in \overline{\Omega}} u(z), \min_{z \in \overline{\Omega}} u(z)$$

are attained only on $\partial\Omega$.
Theorem: Liouville's Theorem
concerns behaviour of non-constant harmonic functions that are defined on as large as a domain possible. (i.e on $\Omega = \mathbb{R}^2$)
Lemma: Harnack's Inequality
Let u be harmonic on $D = D(z_0; R), R > 0$ such that $u(z) \geq 0, \forall z \in D$.
Then, $\forall z \in D$,

$$0 \leq u(z) \leq \left(\frac{R}{R - |z - z_0|}\right)^2 u(z_0)$$

$\left(\frac{R}{R-|z-z_0|}\right)^2$: Positive for any **fixed** $z \in D$, but it goes to ∞ as $z \to \partial D$.
Proof:
$z \in D$ is fixed.
Let $D' = \{w \in \mathbb{R}^2 : |w - z| < R - |z - z_0|\}$
Apply solid MVP to u on D'.

$$0 \leq u(z) = \frac{1}{\pi \left(R - |z - z_0|\right)^2} \iint_{D'} u \, dA$$

Since $u \geq 0$ on D, and $D' \subseteq D$, we know

$$\iint_{D'} u \, dA \leq \iint_{D} u \, dA$$

$$0 \leq u(z) \leq \frac{1}{\pi \left(R - |z - z_0|\right)^2} \iint_{D} u \, dA$$

Now, apply solid MVT again to u on D this time:

$$= \frac{1}{\pi \left(R - |z - z_0|\right)^2} \pi R^2 u(z_0)$$

Theorem: Liouville's Theorem
Let $u \in C^2\left(\mathbb{R}^2\right)$ be harmonic on the entire plane $\mathbb{R}^2$. (u is called an entire harmonic function)
If u is either bounded above on $\mathbb{R}^2$ or bounded below on $\mathbb{R}^2$, then u is constant.
This means $\exists c \in \mathbb{R}$ such that $u(z) \leq c, \forall z \in \mathbb{R}^2$.
Before proving it,
Corollary:
A non-constant entire harmonic function u is neither bounded above nor below.
Hence, it assumes all possible real values. (Image $u(\mathbb{R}^2) = \{u(z) : z \in \mathbb{R}^2 = \mathbb{R}\}$)
Proof:
Suppose $u(z) \leq c, \forall z \in \mathbb{R}^2$
Let $v(z) = u(z) - c, v(z) \geq 0, \forall z \in \mathbb{R}^2$
and v is entire harmonic.
Let $z_0, z_1 \in \mathbb{R}^2$, let $R > |z_0 - z_1|$.
Apply Harnack to v on $D(z_0; R)$.

$$0 \leq v(z_1) \leq \frac{R^2}{(R - |z_0 - z_1|)^2} v(z_0)$$

For any $R > |z_0 - z_1|$.
Let $R \to \infty$ (because v is entire harmonic), we get $v(z_1) \leq v(z_0)$.
Interchanging Roles:

$$v(z_0) \leq v(z_1)$$

So

$$v(z_0) = v(z_1)$$

$\Rightarrow v$ is constant.
Complex Numbers (Review)

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\} \cong \mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}^2\}$$

This bijection allows us to define the structure of a 2-dimentional real vector space on $\mathbb{C}$
That is, we **define** addition and real scalar multiplication on $\mathbb{C}$ by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

(Related to ordered pairs)

$$t \in \mathbb{R}^2, t(a + ib) = (ta) + i(tb)$$

Explicitly, let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$

$$(x, y) = xe_1 + ye_2$$

Let $1 = 1 + 0i, (a = 1 \in \mathbb{R}, b = 0 \in \mathbb{R})$
$i = 0 + 1 \cdot i, (a = 0 \in \mathbb{R}, b = 1 \in \mathbb{R})$
Map $e_1 \mapsto 1, e_2 \mapsto i$ is a real vector space isomorphism from $\mathbb{R}^2 \to \mathbb{C}$.
$\mathbb{C}$ has additional structure.
We have multiplication of 2 elements of $\mathbb{C}$ to give an element of $\mathbb{C}$.
We'll define $(a + ib) \cdot (c + id)$ by demanding that multiplication distributes over addition and that $i \cdot i = -1 = -(1 + 0i)$.

$$\begin{aligned}
(a + ib)(c + id) &= a(c + id) + ib(c + id) \\
&= ac + iad + ibc - bd \quad (i^2 = -1) \\
&= (ac - bd) + i(ad + bc)
\end{aligned}$$

From this definition, we get:

$$z = a + ib, w = c + id, z, w \in \mathbb{C}$$

We get

$$zw = wz$$

Commutative.

$$(z_1 + z_2)w = z_1 w + z_2 w$$
$$z(w_1 + w_2 = zw_1 + zw_2)$$

Distributive.

$$z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$$

If $t \in \mathbb{R}$,
$$(tz)w = z(tw) = t(zw)$$

$$z, w, u \in \mathbb{C}$$
$$(zw)u = z(wu)$$

Associative.

$$1\,(c + id) = c + id$$
$$1 \cdot z = z, \forall z \in \mathbb{C}$$

1 is a multiplicative identity.
Define $\mathbb{R} = \{a + 0i : a \in \mathbb{R}\}$.
(This is not just a one-dimentional vector subspace, it is also closed under multiplication)

$$(a + 0i)\,(c + 0i) = (ac) + 0 \cdot i$$

So
$$\mathbb{R} = \{a + 0 \cdot i : a \in \mathbb{R}\}$$
is a subalgebra of the real algebra $\mathbb{C}$.

$$i\mathbb{R} = \{0 + ib : b \in \mathbb{R}\}$$

is a 1-dimentional vector space, but it is not a subalgebra.

$$(b \cdot i)(d \cdot i) = -bd \notin i\mathbb{R}$$

if b, d both non-zero.
If $z = a + ib$,
$$a = \Re(z) = \mathrm{Re}(z) = \text{ real part of } z$$
$$b = \Im(z) = \mathrm{Im}(z) = \text{ imaginary part of } z$$
Given $z = a + ib$, we define
$$\overline{z} = a - ib$$
This is called the **complex conjugate** of z.
$z \mapsto \overline{z}$ is reflection across x-axis hence it is a real linear isomorphism.

$\overline{tz} = t\overline{z}$ if $t \in \mathbb{R}$
$\overline{z+w} = \overline{z} + \overline{w}$
Claim: $\overline{zw} = \overline{z}\,\overline{w}$
Proof: Directly from definition
$z\overline{z} = a^2 + b^2$
$z\overline{z}$ is real and non-negative and $z\overline{z} = 0$ iff $z = 0$
Define the **modulus** of z to be

$$\begin{aligned}|z| &= \sqrt{z\overline{z}}\\ &= \sqrt{a^2+b^2}\end{aligned}$$

If $z = a + ib$ $|z| \geq 0$ with equality $\iff z = 0$
Claim: $|zw| = |z| \cdot |w|$
Proof:

$$\begin{aligned}|zw|^2 &= (zw)\,\overline{(zw)}\\ &= zw\overline{z}\,\overline{w}\\ &= ...\\ &= |z|^2|w|^2\end{aligned}$$

Corollary:
If $z \neq 0$, then

$$\frac{\overline{z}}{|z|^2}z = z\frac{\overline{z}}{|z|^2} = 1$$

Hence, any **nonzero** $z \in \mathbb{C}$ has a unique multiplicative inverse.

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Remark: Suppose $\mathbb{R}^n$ is given the structure of a real algebra such that

$$|pq| = |p||q|, \forall p, q \in \mathbb{R}^n, |p|^2 = \sum_{i=1}^{n}(p_i)^2$$

And such that any non-zero $p \in \mathbb{R}^n$ has a multiplicative inverse (Division algebra).
Do not assume commutative.
Do not assume associative.
Theorem: Hurwitz 1898.
Only Four.
Quaternion.
Octonions.

9 January 24th

Note:
If $z = a + ib \iff (a, b) \in \mathbb{R}^2$, then $iz = ia - b \iff (-b, a) \in \mathbb{R}^2$.
Multiplication by i is $90°$ counterclockwise roration.
Geometric Interpretation.
$z = (x, y) \in \mathbb{R}^2$
$D(z; R) = \{w \in \mathbb{R}^2 : |w - z| < R\}$
$z = x + iy, |z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$
$z \in \mathbb{C}, D(z; R) = \{w \in \mathbb{C} : |z - w| < R\}$
$\mathbb{C}$ and $\mathbb{R}^2$ have the same topology.
From now on, we identify $\mathbb{R}^2$ and $\mathbb{C}$.
Definition:
Let $\Omega \subseteq \mathbb{C}$ be a subset, (usually Ω will be a domain).
A complex-valued function on Ω is a map $f : \Omega \to \mathbb{C}, \forall z \in \Omega, w = f(z) \in \mathbb{C}$.
Examples:

1. $f(z) = z^2$
2. $g(z) = \overline{z}$
3. $h(z) = \frac{1}{1-z}$

The examples above have different domains.

$$\mathbb{C} \ni z = x + iy \iff (x, y) \in \mathbb{R}^2$$
$$u + iv = w : f(z) = f(x, y)$$

Hence

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$
$$u(x, y) = \mathrm{Re}\,(f(x, y))$$
$$v(x, y) = \mathrm{Im}\,(f(x, y))$$

So a complex-valued function on Ω is equivalent to two real-valued functions on Ω.
Examples:

$$z = x + iy$$
$$f(z) = z^2 = (x + iy)(x + iy) = (x^2 - y^2) + (i2xy)$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

$$g(z) = \overline{z} = x - iy$$
$$u(x, y) = x$$

$$v(x,y) = -y$$

$$h(z) = \frac{1}{1-z}\left(\frac{1-\overline{z}}{1-\overline{z}}\right) = \frac{1-\overline{z}}{|1-z|^2} = \frac{1-(x-iy)}{(1-x)^2+y^2}$$

$$u(x,y) = \frac{1-x}{(1-x)^2+y^2}$$

$$v(x,y) = \frac{y}{(1-x)^2+y^2}$$

Remark:
Let $f : \Omega \to \mathbb{C}$.
Graph of $f = \{(z, f(z)) \in \mathbb{C}\times\mathbb{C}, z \in \Omega\}$ $\{(x, y, u(x, y), v(x, y)) \in \mathbb{R}^4; (x, y) \in \Omega\}$ is a subset of $\mathbb{R}^4$. So we can't draw it.
Limits
Let Ω be open subset of $\mathbb{C}$ and let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$. ($f$ need not be defined at z_0)
We say that $\lim_{z\to z_0} f(z) = w_0$ iff $\forall\epsilon > 0, \exists\delta > 0$ such that if $z \in \Omega \cap D(z_0; \delta)$, then $f(z) \in D(w_0; \epsilon)$.
Since $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and the notion of open sets is the same in $\mathbb{R}^2$ and $\mathbb{C}$.
This is exactly the definition of limits from Calc 3.
Let $w_0 = u_0 + iv_0$
$f(z) = u(x, y) + iv(x, y)$

$$|w_0 - f(t)| = \sqrt{\left(u(x,y)-u_0\right)^2 + \left(v(x,y)-v_0\right)^2}$$

$f(z) \to w_0 \iff u(x, y) \to u_0$ AND $v(x, y) \to v_0$

$$\lim_{z\to z_0} f(z) = w_0 \iff \begin{cases} \lim_{z\to z_0} u(x,y) = u_0 \\ \lim_{z\to z_0} v(x,y) = v_0 \end{cases}$$

Lemma:
Suppose $\lim_{z\to z_0} f_1(z) = w_1, \lim_{z\to z_0} f_2(z) = w_2$.
Then,

1. $\lim_{z\to z_0} (f_1(z) \pm f_2(z)) = w_1 \pm w_2$

   Automatic.

   Proposition:

2. $$\lim_{z\to z_0} f_1(z)f_2(z) = w_1w_2$$

3. $$\lim_{z\to z_0} \frac{f_1(z)}{f_2(t)} = \frac{w_1}{w_2}$$

   provided $w_2 \neq 0$ and $f_2(z) \neq 0$ in a neighbourhood of z_0.

Proof:
(b)
$$f_1(z) = u_1(x,y) + iv_1(x,y)$$
$$f_2(z) = u_2(x,y) + iv_2(x,y)$$

$$\begin{aligned} f_1(z)f_2(z) &= (u_1(x,y)u_2(x,y) - v_1(x,y)v_2(x,y)) \\ &+ i\,(u_1(x,y)v_2(x,y) + v_1(x,y)u_2(x,y)) \\ &= w_1 w_2 \end{aligned}$$

For (c)
$$\frac{u_1 + iv_1}{u_2 + iv_2} = \frac{(u_1 + iv_1)(u_2 - iv_2)}{u_2^2 + v_2^2} = \dots$$

Continuity
Let $\Omega \subseteq \mathbb{C}$ be open.
$f : \Omega \to \mathbb{C}$.
Let $z_0 \in \Omega$.
We say f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$
Notice that $z_0 = (x_0, y_0), f(z) = u(x,y) + iv(x,y)$
f is continuous at z_0 iff both u and v are continuous at (x_0, y_0)
We say f is continuous on Ω if f is continuous at z for all $z \in \Omega$.
It is clear from the properties of limits, that if f and g are continous at z_0. Then so is $f \pm g, fg, \frac{f}{g}$, provided $g(z) \neq 0$ in a neighbourhood of z_0.
Corollary:
Since $f(z) = z$ and $g(z) = c =$ constant are clearly continuous everywhere, it follows that the polynomials

$$a_0 + a_1 z + \cdots + a_n z^n$$

are continuous everywhere.
And rational functions,
$$\frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n}$$
are continuous everywhere where denominator is non-zero.
Suppose
$$f : U \subseteq_{\text{open}} \mathbb{C} \to \mathbb{C}$$
$$g : V \subseteq_{\text{open}} \mathbb{C} \to \mathbb{C}$$
with $f(u) \subseteq V$.
Then $h = g \circ f : U \to \mathbb{C}$ if f is continuous at $z_0 \in U$ and g is continuous at $f(z_0) \in V$.
Then $g \circ f$ is continuous at $z_0 \in U$.
Proof:
You have already done this.
The complex derivative
This is new and different.

Let $\Omega \subseteq_{\text{open}} \subseteq \mathbb{C}$.
Let $f : \Omega \to \mathbb{C}$, let $z_0 \in \Omega$.
We say that f is **complex differentiable** at z_0 iff.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.
Involves being able to devide by non-zero elements of $\mathbb{R}^2 \cong \mathbb{C}$.
We can't do this in $\mathbb{R}^n$, $n > 2$.
For z sufficiently close to z_0, since Ω is open.

$$\frac{f(z) - f(z_0)}{z - z_0}$$

is a function of z_0.
The notion of existence of limit is the usual multivariable calculus notion.
Example:
$f(z) = z^2, \Omega = \mathbb{C}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0}$$
$$= \lim_{z \to z_0} (z + z_0) = 2z_0$$
$$f'(z) = 2z, \forall z \in \mathbb{C}$$

Lemma:
Suppose f is complex differentiable at z_0.
Then f is continuous at z_0.
Proof:
$f(z) - f(z_0)$ is equal to $\frac{f(z) - f(z_0)}{(z - z_0)}(z - z_0) \to 0$
Properties of the complex derivative
Suppose f and g are differentiable at z.

$$(f \pm g)'(z) = f'(z) \pm g'(z)$$
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

$g(z) \neq 0$
Enough by continuity.
Proof:
The proof is the same as in Calculus 1.
Because we have the same limit laws and that's all you need.

10 January 27th

From last time:
Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}, z_0 \in \Omega$.
We say f is (complex) differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

The Chain Rule
Let $f : U \to \mathbb{C}, g : V \to \mathbb{C}$ such that $f(U) \subseteq V$.
$h = g \circ f : U \to \mathbb{C}$ if f is differentiable at z_0 and g is differentiable at $f(z_0)$.
Then $h = g \circ f$ is differentiable at z_0, and

$$h'(z_0) = g'(f(z_0))f'(z_0)$$

Proof:
Let $w_0 = f(z_0) \in V$. Define a map $T : V \to \mathbb{C}$ by

$$T(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0)$$

for $w \neq w_0$.
$T(w_0) = 0$ by construction, T is continuous at w_0.
Solve for $g(w) - g(w_0)$.

$$g(w) - g(w_0) = (g'(w_0) + T(w)) \cdot (w - w_0)$$

also works when $w = w_0$.
The equation is true for all $w \in V$.
Let $w = f(z), z \in U, w_0 = f(z_0)$. Divide both sides by $z - z_0$.
$z \neq z_0$

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = [g'(f(z_0)) + T(f(z))] \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

$T(f(z)) \to 0$ by continuity of f at z_0, T at w_0.
Analytic Functions
Define:
$\Omega \subseteq_{\text{open}} \mathbb{C}$. Let $f : \Omega \to \mathbb{C}$.
Let $z_0 \in \Omega$.
We say f is **analytic** (also called complex analytic, also called **holomorphic**) at z_0 if $f'(z)$ exists and is continuous in some open neighbourhood $U \subseteq \Omega$ of z_0.
Notice:
By its definition, f is **analytic** at $z_0 \in \Omega$ iff f is analytic at all points in some open neighbourhood of z_0.
Examples:

$\exists U$ open, $U \subseteq \Omega$, $U \ni z_0$ such that f' exists and is continuous on U.
So f is analytic at $z \forall z \in U$.
Remark:
It is actually true that if f is (complex) differentiable on an open set U, then its (complex) derivative f' must be continous on U.
But this is a hard theorem. (Goursat's Theorem).
We will do it later in course.
Examples:

1. Any polynomial is analytic on $\mathbb{C}$.
2. Non-example:

   Consider the map $g : \mathbb{C} \to \mathbb{C}, g(z) = z\overline{z} = |z|^2 = u(x, y) + iv(x, y)$

   $v(x, y) = 0, u(x, y) = x^2 + y^2$.

   On Assignment 2: You show that g is **complex** differentiable at $(0, 0)$ but nowhere else.

   So this function g is nowhere analytic.

   This $g \in C^\infty\left(\mathbb{R}^2\right)$. So as a function $g : \mathbb{R}^2 \to \mathbb{R}^2$.

   It is differentiable everywhere in sense of Calc 3.

   But it is not complex differentiable except at origin.

Cauchy-Riemann Equations
Theorem:
Let $f : \Omega \subseteq_{\text{open}} \mathbb{C} \to \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$.
Then, u_x, u_y, v_x, v_y all exist at z_0 and

$$u_x(z_0) = v_y(z_0)$$

$$u_y(z_0) = -v_x(z_0)$$

Proof:
$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists by assumption.
We get the same limiting value regardless of what path we take in the plane Ω to (x_0, y_0).
Consider the path $y = y_0, x \to x_0$.
See pictures.
Remark:
The converse in not true. That is, suppose u_x, u_y, v_x, v_y all exist at z_0 and satisfy (*).
Then f does not have to be (complex) differentiable at z_0).
(Counterexample on A2).
You already saw something like this in Calc 3. Suppose $F : U \subset_{\text{open}} \mathbb{R}^n \to \mathbb{R}^m$ if F is differentiable at $\vec{a} \in U$, then $\frac{\partial F_i}{\partial x_i}(\vec{a})$ exists $\forall i = 1, ..., m, j = 1, ..., m$
But one can have that all first partial derivatives exist at $\vec{a} \in U$ and F still not be differentiable at $\vec{a}$.

Stronger Assumption: if $F \in C^1(u)$, then F is differentiable on U.
We have a similar result here.
Theorem:
Let $\Omega \subseteq \mathbb{C}$ be open. Let $u, v \in C^1(\Omega)$ (u_x, u_y, v_x, v_y exist and are continuous on all of Ω)
Suppose $u_x = v_y$ and $u_y = -v_x$ everywhere on Ω.
Then f is analytic on Ω.
Proof:
$u, v \in C^1(\Omega) \to u$ and v are differentiable on Ω in the sense of Calculus 3.
Let $z_0 \in \Omega$.

$$\begin{aligned} u(z) - u(z_0) &= u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + Q_1(x, y) \\ v(z) - v(z_0) &= v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) + Q_2(x, y) \end{aligned}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{Q_k(x, y)}{||(x, y) - (x_0, y_0)||} = 0, k = 1, 2$$

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(z) + iv(z) - (u(z_0) + iv(z_0))}{z - z_0} \\ &= \frac{u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + Q_1(x, y)}{z - z_0} \\ &+ i\frac{v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) + Q_2(x, y)}{z - z_0} \\ &= \frac{u_x(z_0)(z - z_0) + iv_x(z_0)(z - z_0) + Q_1(x, y) + Q_2(x, y)}{z - z_0} \\ &= u_x(z_0) + iv_x(z_0) + \frac{Q_1(x, y)}{z - z_0} + \frac{Q_2(x, y)}{z - z_0} \end{aligned}$$

Let $z \to z_0$.

$$|\frac{Q(x, y)}{z - z_0}| = \frac{|Q(x, y)|}{||(x, y) - (x_0, y_0)||}$$

goes to 0 as $z \to z_0$.
We have shown that $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z \in \Omega$.
So f is differentiable on Ω.
We have also shown that

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z), u, v \in C^1(\Omega)$$

So f' is continuous on Ω. So f is analytic on Ω.

$$u_x = v_y, u_y = -v_x$$

Cauchy-Riemann Equations.

11 January 29th

From last time:

$$\Omega \subseteq_{\text{open}} \mathbb{C}$$

.
$f : \Omega \to \mathbb{C}$ is analytic or Ω if f' exists and is continuous on Ω.
Let $f = u + iv$.
Theorem:
with $u, v \in C^1(\Omega)$, then f is analytic on Ω iff $u_x = v_y, u_y = -v_x$. (Cauchy-Riemann equations) on Ω.
Example:
$f(z) = z^2 = (x^2 - y^2) + i2xy$
$u(x, y) = x^2 - y^2, v(x, y) = 2xy$
Cauchy-Riemann equations are satisfied.
$f'(z) = u_x + iv_x = v_y - iu_y$
Suppose we consider $u(x, y) + iv(x, y)$ where $u(x, y) = x^2 - y^2, v(x, y) = c \cdot xy$.
$u_x = 2x, u_y = -2y, v_x = cy, v_y = cx$
Analytic iff $c = 2$.
Complex analyticity is much more "rigid" than real variable differentiability.
Cauchy-Riemann equations in polar corrdinates
$f(z) = u(x, y) + iv(x, y) = u(r, \theta) + iv(r, \theta)$
Polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

for $r > 0$.
Suppose $h \in C^1(\mathbb{R}^2 \setminus \{0\})$ by Chain rule,

$$h_r = h_x \frac{\partial x}{\partial r} + h_y \frac{\partial y}{\partial r} = \cos\theta h_x + \sin h_y$$

$$h_\theta = h_x \frac{\partial x}{\partial \theta} + h_y \frac{\partial y}{\partial \theta} = -r\sin\theta h_x + r\cos\theta h_y$$

$$\begin{aligned} u_r &= \cos\theta u_x + \sin\theta u_y \\ &= \cos\theta v_y - \sin\theta v_x \\ &= \frac{1}{r} v_\theta \end{aligned}$$

$$\begin{aligned} v_r &= \cos\theta v_x + \sin\theta v_y \\ &= -\cos\theta u_y + \sin\theta u_x = -\frac{1}{r} u_\theta \end{aligned}$$

Check the other direction, we conclude that

$$u_x = v_y, u_y = -v_x \iff u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$$

Example:
$\Omega : \mathbb{C} \setminus \{(x, 0) : x \leq 0\}$
Define $f : \Omega \to \mathbb{C}$ by $f(z) = \log r + i\theta$
$\theta = \arg(z), -\pi < \theta < \pi$.
$u(r, \theta) = \log r, v(r, \theta) = \theta$.

$$u_r = \frac{1}{r} \qquad u_\theta = 0$$
$$v_r = 0 \qquad v_\theta = 1$$

$u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$.
Hence,

$$\log r + i\theta$$

is analytic on Ω.
You can also check this in x, y coordinates.

$$u(x, y) = \log \sqrt{x^2 + y^2}, v(x, y) = \arg(x, y)$$

This example is very important and will come back on Friday.
Relation between analytic $\mathbb{C}$-valued functions and harmonic $\mathbb{R}$-valued functions
Let $f : \Omega \to \mathbb{C}$ by analytic on Ω.
And suppose $u, v \in C^2(\Omega)$. [In fact, this is always true. We will prove this later.]

$$u_x = v_y, u_y = -v_x$$
$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx}$$
$$u_{yy} = (u_y)_y = -(v_x)_y = -v_{xy}$$

Since $v \in C^2(\Omega), v_{xy} = v_{yx} \Rightarrow u_{xx} + u_{yy} = 0$.
Thus, u is harmonic.
Similarly, v is harmonic.
So we have shown that the real and imaginary parts of analytic function are harmonic. (Modulo the assumption that we'll get rid of later)
Exponential Functions
Recall from Calculus 1, we have a function $\exp : \mathbb{R} \to \mathbb{R}, \exp(x) = e^x$
Properties:
exp is C^∞, $\exp(0) = 1$.
$\exp' = \exp, \exp > 0\ \forall x$
$\exp(x + y) = \exp(x)\exp(y)$

$$\exp(-x) = [\exp(x)]^{-1}$$

We seek a function $f : \mathbb{C} \to \mathbb{C}$ such that f is differentiable and $f'(z) = f(z)\ \forall z \in \mathbb{C}$.
This implies:

- f differentiable $\to$ f continuous on $\mathbb{C}$
- $f' = f \to f'$ continuous on $\mathbb{C}$. Hence f must be **analytic** on $\mathbb{C}$.

So $f = u + iv$ must satisfy Cauchy-Riemann equations.

$$f' = u_x + iv_x = v_y - iu_y = f = u + iv$$

$$\to u_x = u \to u(x, y) = a(y)e^x$$

for some C^∞ function $a(y)$.

$$v_x = -u_y = v \to v(x, y) = b(y)e^x$$

for some C^∞ function $b(y)$.

$$u_x = ae^x = v_y = e^x b'$$
$$v_x = be^x = -u_y = -e^x a'$$
$$a = b', b = -a'$$
$$b'' = a' = -b \to b(y) : C\cos(y) + D\sin(y)$$
$$a'' = -b' = -a \to a(y) = A\cos(y) + B\sin(y)$$

We also want $f(0) = 1$ just like in the real case.

$$u(0, 0) = 1 = a(0)e^0 = a(0) = A$$

$$v(0, 0) = 0 = b(0)e^0 = b(0) = -B$$

$$B = 0, A = 1$$

We have shown that

$$f(z) = e^x \cos(y) + ie^x \sin(y) = \exp(z)$$

has following properties. f is analytic on $\mathbb{C}$, $f'(z) = f(z)\ \forall z \in \mathbb{C}, f(0) = 1$
This is called the complex exponential function.
Observer: if $z = x$ is real $(y = 0)$.
Then $\exp(z) = e^x = \exp(x)$.
So the restriction of exp to the real line gives the real exp function.
It is no longer true that $\exp(z) > 0$.
Example:

$$\exp(\pi i) = (x = 0, y = \pi) = -1$$
$$\exp(\frac{\pi}{2} i) = i$$
We will show soon that $\exp(z) \neq 0 \; \forall z$ and in fact
$$\exp(\mathbb{C}) = \{\exp(z) : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}$$
Periodicity:

$$\begin{aligned} &\exp\left(z + 2\pi i k\right), k \in \mathbb{Z} \\ =&\exp(z) \end{aligned}$$

$\exp(z)$ is periodic with period $2\pi i$ (very different from real case).

$$|\exp(z)|^2 = |e^x \cos(y) + i e^x \sin(y)|^2$$
$$\left(e^x \cos y\right)^2 + \left(e^x \sin y\right)^2 = e^{2x} > 0, \forall (x, y) \Rightarrow \exp(z) \neq 0 \; \forall z$$

$$\exp(2\pi i) = \exp(0) = 1$$
Let $z_1 = x_1 + i y_1, z_2 = x_2 + i y_2$.

$$\begin{aligned} \exp(z_1 + z_2) &= \exp\left((x_1 + x_2) + i(y_1 + y_2)\right) \\ &= e^{x_1 + x_2} \cos(y_1 + y_2) + i e^{x_1 + x_2} \sin(y_1 + y_2) \\ &= e^{x_1} e^{x_2} \left(\cos y_1 \cos y_2 - \sin y_1 \sin y_2\right) + i e^{x_1} e^{x_2} \left(\sin y_1 \cos y_2 + cos y_1 \sin y_2\right) \\ &= e^{x_1} \left(\cos y_1 + i \sin y_1\right) \cdot e^{x_2} \left(\cos y_2 + i \sin y_2\right) \\ &= \exp(z_1) \exp(z_2) \end{aligned}$$

$$\exp(z_1 + z_2) = \exp(z_1) \exp(z_2) \; \forall z_1, z_2 \in \mathbb{C}$$
Definition:
Let $e \in \mathbb{R}$ be the usual base of natual logarithm.
Let $w \in \mathbb{C}$
Define:

$$e^w := \exp(w)$$
$z = x + iy$ $e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$
$e^{iy} = \cos(y) + i \sin(y) \; \forall y \in \mathbb{R}$
Euler's formula.
(Plug in $x = 0$ into $\exp(z)$)
Corollary:

$$e^{i\pi} = -1$$

12 January 31st

From last time:
Let $z = x + iy \in \mathbb{C}$.

$$\exp(z) = e^z = e^x \cos(y) + ie^x \sin(y)$$

exp is analytic on $\mathbb{C}$.

$$\frac{\partial}{\partial z} e^z = e^z \ \forall z$$

$$e^z \neq 0 \ \forall z$$

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

$$e^{z+2\pi ik} = e^z \ k \in \mathbb{Z}$$

$$e^{2\pi ik} = 1 \ k \in \mathbb{C}$$

For $y \in \mathbb{R}$,

$$e^{iy} = \cos(y) + i\sin(y), |e^{iy}| = 1$$

Polar form of a complex number
$\mathbb{C} \ni z = x + iy \iff (x, y) \in \mathbb{R}^2$ in polar coordinates: $x = r\cos\theta, y = r\sin\theta, r = \sqrt{x^2 + y^2} = |z|$.

$$\begin{aligned} z &= r\cos\theta + r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$$z = re^{i\theta}$$

Polar form of z.

$$|z| = |re^{i\theta}| = |r||e^{i\theta}| = r$$

θ is only defined modulo integer multiplies of 2π.
$\tilde{\theta} = \theta + 2\pi k, k \in \mathbb{Z}$ also works.

$$\begin{aligned} re^{i\tilde{\theta}} &= re^{i(\theta+2\pi k)} \\ &= re^{i\theta}e^{i2\pi k} = re^{i\theta} = 1 \end{aligned}$$

$e^{i\theta}$ is called the phase of $z = \frac{z}{|z|}$ for $z \neq 0$.
Suppose $z_1, z_2 \neq 0$,

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

(Defined up $\mathbb{Z}$-multiples of 2π).
Let $z = re^{i\theta} = x + iy$
$\overline{z} = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$
Everything works.

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{re^{-i\theta}}{r^2} = \frac{1}{r}e^{-i\theta}$$

Application of polar form:
nth power + nth roots

$$z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$$

n is positive integer.
Example:
$z = 1 + \sqrt{3}$
$1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}$

$$\left(1 + \sqrt{3}i\right)^3 = 2^3 e^{\left(\frac{i\pi}{3}\right)} = -8$$

nth roots
Let $z = re^{i\theta} = re^{i(\theta + 2k\pi)}, k \in \mathbb{Z}$
Claim:

$$\zeta_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}, k = 0, 1, \ldots, n-1$$

are n distinct nth roots of z.
(We know that $\zeta^n = z$ has at most n distinct roots from algebra.)

$$\zeta^n - z = (\zeta - \zeta_0)(\zeta - \zeta_1)\ldots(\zeta - \zeta_{n-1})$$

First we show that $(\zeta_k)^n = z$.

$$\begin{aligned}(\zeta_k)^n &= \left[r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}\right]^n \\ &= re^{i\theta} = z\end{aligned}$$

Suppose $0 \leq j < k \leq n-1$ need to show $\zeta_j \neq \zeta_k$.
Suppose $\zeta_k = \zeta_j$

$$1 = \frac{\zeta_k}{\zeta_j} = e^{i(2\pi)\frac{k-j}{n}}$$

$$0 < \frac{k-j}{n} < \frac{n-1}{n} < 1$$

Contradiction
Examples:
$z = 1, (r = 1)$.

nth roots of unity $= \{e^{\frac{2\pi ik}{n}}, k = 0, 1, \ldots, n-1\}$
Pictures here.
The Logarithm

- We want to define an **inverse** to the exponential function (which will be the logarithm). However, $\exp : \mathbb{C} \to \mathbb{C}$ is **NOT** one-to-one.

Theorem:
Let $S = \{z = x + iy : -\pi < y \leq \pi\}$, then exp maps S bijectively onto $\mathbb{C} \setminus \{0\}$. And exp maps the line $y = \pi$ onto the negative x-axis.
Proof:
We already know that $\exp(z) \neq 0$ for all z.
It is clear that $\exp(x + i\pi) = e^x e^{i\pi} = -e^x$ is on negative x-axis.
Let $w \neq 0$, we want to show $\exists! z \in S$ with $\exp(z) = w$.
Let $w = ge^{i\phi}$ in polar form, $g > 0, -\pi < \phi \leq \pi$.
Let $x = \log g, y = \phi$.
$x + iy \in S$,

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^{\log g + i\phi} \\ &= e^{\log g} e^{i\theta} = ge^{i\phi} = w \end{aligned}$$

Suppose $e^{z_1} = e = e^{z_2}$ for $z_1, z_2 \in S$.

$$e^{z_1 - z_2} = 1 \iff z_1 - z_2 = 2\pi ik, k \in \mathbb{Z}$$

z_1, z_2 are an the same vertical line.
$|z_1 - z_2| = 2\pi k$ both in S, on same vertical line

$$|z_1 - z_2| < 2\pi \Rightarrow k = 1 \Rightarrow z_1 = z_2$$

Remark:
We could also take

$$S_b = \{x + iy : b - 2\pi < y \leq b\}$$

for any real number b.
(We took $b = \pi$)
In the same way, exp maps S_b bijectively onto $\mathbb{C} \setminus \{0\}$ with the upper edge of S_b mapped onto the ray $\phi = b$ in w-plane.
We can now define (infinitely many) "inverses" to exp.
Definition: Let $z \neq 0$, we can write $z = re^{i\theta}, r > 0$.
Choose a strip $b - 2\pi < \theta \leq b$.
Define $\log(z) = \log(r) + i\theta$.
Where θ is the unique argument of z in the range $b - 2\pi < \theta \leq b$.

$$e^z = e^{\log r + i\theta} = e^{\log r} e^{i\theta} = re^{i\theta} = z$$

So we get many different "branches" of the logarithm. (Which is a multi-valued function)

$$S_b \rightleftharpoons \mathbb{C} \setminus \{0\}$$

Inverse to each other.
When $b = \pi$, we usually call this the principal branch of logarithm.
This branch $\log z$ is not continuous on the ray $\phi = b$ in w-plane.
To avoid this problem, we remove this ray from the domain.

$$S_b^0 = \{x + iy \in \mathbb{C}; b - 2\pi < y < b\}$$

exp maps S_b^0 bijectively onto $\mathbb{C} \setminus \{0\} \setminus \{\text{ray}\phi = b\}$ and the function

$$re^{i\theta} = z \mapsto \log(z) = \log(r) + i\theta$$

with $b - 2\pi < \theta < b$ is an inverse to exp and **is** continuous on its domain.
Next time:
We will show that the branches of logarithm are **analytic** and compute the complex derivative.

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From last time:
$b \in \mathbb{R}$,

$$S_b = \{x + iy : b - 2\pi < y \leq b\}$$

exp maps S_b bijectively onto $\mathbb{C} \setminus \{0\}$

$$S_b^0 = \{x + iy : b - 2\pi < y < b\}$$

exp maps S_b^0 bijectively onto $\mathbb{C} \setminus \{\text{ray } \phi = b\}$ in the w-plane.
Pictures here.
$z = re^{i\theta}$
$w = \rho e^{i\phi}$
Inverse map is called a branch of the logarithm (which is really a multivalued function).
If $w = \rho e^{i\phi}, \rho > 0, b - 2\pi < \phi < b$,
Define $\log w = \log \rho + i\phi$.
With this restriction to a particular "branch".
The logarithm function is analytic on its domain. (We did this when we talked about CR in polar coordinates)

$$z = re^{i\theta}, r > 0$$

$$f(z) = \log r + i\theta = u(r, \theta) + iv(r, \theta)$$

$$u_r = \frac{1}{r} v_\theta$$
$$v_r = -\frac{1}{r} u_\theta$$

CR equations are satisfied.
θ : Any branch of the argument function.
Any branch of log is analytic on its domain which is $\mathbb{C} \setminus \{0\} \setminus \{\text{ray } \theta = b\}$
Principal branch.
Hence, $\log'(z)$ exists $\forall z \in D_b$.
We seek a formula for derivative of $\log z$.
We know $\exp(\log z) = z, \forall z \in D_b$ because they are inverses of each other.
So by the Chain rule, if I differentiate both sides with respect to z

$$1 = \exp'(\log z) \log'(z) = \exp(\log(z)) \log'(z)$$

Thus, $\log'(z) = \frac{1}{z}$ just like real case. (For any branch).
Addition Law

$$\log(z_1 z_2) = \log z_1 + \log z_2 + 2\pi i k$$

For some $k \in \mathbb{Z}$ which depends on branch choices.
Claim:
Let $z \neq 0, z^{\frac{1}{n}} = \exp\left(\frac{1}{n} \log z\right)$
Set of values because log is multivalued.
We claim that it is exactly the set of n distinct $n^t h$ roots of z.
Proof:

$$z = re^{i\theta} = re^{i(\theta + 2\pi k)}, r > 0, k \in \mathbb{Z}.$$

$$\log z = (\log r) + i\left(\theta + 2\pi k\right)$$
$$\frac{1}{n} \log z = \frac{1}{n} \log r + i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)$$

$$\begin{aligned}
\exp(\frac{1}{n} \log z) &= \exp\left(\frac{1}{n} \log r + i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right) \\
&= \exp\left(\frac{1}{n} \log r\right) \exp\left(i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right) \\
&= r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)} \\
&= z^{\frac{1}{n}}
\end{aligned}$$

This motivates:

Complex exponents.
Let $c \in \mathbb{C}$. We want to define z^c for $z \neq 0$.
In real case, $c \in \mathbb{R}, x > 0$

$$x^c = e^{\log(x^c)} = e^{c \log x} = \exp(c \log x)$$

We do the same:

$$z^c := \exp(c \log z)$$

This is multivalued because log is.

$$\begin{aligned}
i^i &= \exp(i \log i) \\
&= \exp\left(i\left(\frac{2\pi}{n} + 2\pi i k\right)\right), k \in \mathbb{C} \\
&= \exp\left(\frac{-\pi}{2} - 2\pi k\right) \\
&= e^{\frac{-\pi}{2} - 2\pi k}
\end{aligned}$$

All the infinitely many values of i^i are real!

$$\begin{aligned}
\frac{d}{dz}(z^c) &= \frac{d}{dc}\left(\exp\left(c \log z\right)\right) \\
&= \exp'\left(c \log z\right) \cdot \frac{d}{dz}\left(c \log z\right) \\
&= \exp\left(c \log z\right) \cdot \frac{c}{z} = z^c \cdot \frac{c}{z} = cz^{c-1}
\end{aligned}$$

Trignometric Functions
Recall:
If $x \in \mathbb{R}$,

$$e^{ix} = \cos(x) + i\sin(x) \to \cos(x) = \frac{e^{ix} + e^{-iex}}{2}, x \in \mathbb{R}$$

$$e^{-ix} = \cos(x) - i\sin(x) \to \sin(x) = \frac{e^{ix} - e^{ix}}{2i}$$

For any $z \in \mathbb{C}$, define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\begin{aligned}
\frac{d}{dz}\left(\cos z\right) &= \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) \\
&= \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) \\
&= \frac{-1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin(z)
\end{aligned}$$

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$$
$$= \cos(z)$$

Clear:

$$\cos(-z) = \cos(z)$$
$$\sin(-z) = -\sin(z)$$

$$\sin^2(z) + \cos^2(z) = 1$$

This equation gives no bounds on $|\sin(t)|$ or $\cos(t)$.
Disgression: Real hyperbolic functions

$$\cosh, \sinh : \mathbb{R} \to \mathbb{R}$$
$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$
$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$

Defined.

$$\frac{d}{dt}\cosh(t) = \sinh(t)$$
$$\frac{d}{dt}\sinh(t) = \cosh(t)$$
$$\cosh^2(t) - \sinh^2(t) = 1$$

Thus, $\cosh^2(t) - \sinh^2(t) = 1$
Hyperbolic functions.
Circular functions.
Complex Hyperbolic Functions
Define: $\forall z \in \mathbb{C}$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

cosh, sinh are analytic on $\mathbb{C}$.

$$\frac{d}{dz}\cosh(z) = \sinh(z)$$
$$\frac{d}{dz}\sinh(z) = \cosh(z)$$

$$\cosh^2(z) - \sinh^2(z) = 1$$

Go back to compute trig functions.

$$\begin{aligned}\cos(z) &= u(x,y) + iv(x,y) \\ &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \frac{e^{i(x+iy)} + e^{-i(x+y)}}{2} \\ &= \frac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right) \\ &= \frac{1}{2}\left((\cos x + i\sin x)\, e^{y} + (\cos(x) - i\sin(e)e^{y})\right)\end{aligned}$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\begin{aligned}\sin(z) &= -\frac{d}{dz} \\ &= -\left(u_x + iv_x\right)\end{aligned}$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Don't memorize any of these.
Only remember:

$$\begin{aligned}e^{iz} &= \cos(z) + i\sin(z) \\ e^{z} &= \cosh(z) + \sinh(z)\end{aligned}$$

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From last time:

$$\begin{aligned}z &\in \mathbb{C} \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ e^{iz} &= \cos(z) + i\sin(z) \\ \cos(-z) &= \cos(z) \\ \sin(-z) &= -\sin(z)\end{aligned}$$

$$\frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\sin z = \cos z$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$
$$e^z = \cosh(z) + \sinh(z)$$
$$\cosh(-z) = \cosh(z)$$
$$\sinh(-z) = -\sinh(z)$$
$$\frac{d}{dz}\cosh(z) = \sinh(z)$$
$$\frac{d}{dz}\sinh(z) = \cosh(z)$$

All four functions are **analytic** on all of $\mathbb{C}$.

$$\cos(iz) = \cosh(z)$$
$$\sin(iz) = i\sinh(z)$$
$$\cosh(iz) = \cos(z)$$
$$\sinh(iz) = i\sin(z)$$

Special Case:
Let $z = x \in \mathbb{R}$.

$$\sin(ix) = i\sinh(x)$$
$$\cos(ix) = \cosh(x)$$

So cos and sin are unbounded on $\mathbb{C}$ because they are unbounded on the purely imaginary axis.
Similarly, $\sinh(ix) = i\sin(x), \cosh(ix) = \cos(x)$.
cosh and sinh are bounded on imaginary axis.
Summary:
cos and sin periodic in x-direction unbounded in iy-direction.
cosh and sinh unbounded in x-direction. Periodic in iy-direction.
Inverse Trig and Inverse Hyperbolic Functions:
(By one example)
Suppose $z = \cos w$. How can we find w?

$$\cos(w) = \frac{e^{iw} + e^{-iw}}{2}$$
$$2z = e^{iw} + e^{-iw}$$
$$e^{iw} - 2z + e^{-iw} = 0$$

$$\left(e^{iw}\right)^2 - 2z\left(e^{iw}\right) + 1 = 0$$
$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$
$$iw = \log\left(z \pm \sqrt{z^2 - 1}\right)$$
$$w = -i\log\left(z \pm \sqrt{z^2 - 1}\right) = \cos^{-1}(z)$$

Multivalued because log is multivalued.

$$\cos^{-1}(i) = -i\log\left(i \pm \sqrt{-2}\right) = -i\log\left(i \pm \sqrt{2}i\right)$$

Harmonic Conjugates
Recall:
If $f = u + iv$ is **analytic** on Ω (and if $u, v \in C^2(\Omega)$, (We will see that it is always true), then u, v are harmonic on Ω.
Question, suppose $u \in C^2(\Omega)$ is harmonic on Ω.
Can we find a $v \in C^2(\Omega)$ harmonic on Ω such that $f = u + iv$ is analytic.
If such v exists, then v is called a **harmonic conjugate** of u.
Note: By our definition of analytic, v must be at least C^1, the Cauchy-Riemann equations must be satisfied:

$$u_x = v_y, u_y = -v_x$$
$$u \in C^2 \to u_x, u_y \in C^1$$
$$v_x, v_y \in C^1 \Rightarrow v \in C^2$$

Suppose $v, \tilde{v}$ are **both** harmonic conjugate of u

$$(v - \tilde{v})_x = 0$$
$$(V - \tilde{v})_y = 0$$

If Ω is connected.

$$v - \tilde{v} \text{ is constant}$$

It is clear that if v is a harmonic conjugate of u, then $v+c$ is also, for any constant $c \in \mathbb{C}$, we just showed that if Ω is connected, this the entent of non-uniqueness.
What about existence?
Suppose $v \in C^2(\Omega)$, let $F = \nabla v = (v_x, v_y)$ is a C^1 vector field or Ω.
From the Fundamental theorem of line integrals,

$$\int_{z_0}^{z} F \cdot dr = v(z) - v(z_0)$$

for any path from z_0 to z lying entirely in Ω.
We start with $u \in C^2(\Omega)$, which is harmonic on Ω. We seek a harmonic conjugate v.

We must have

$$v_x = -u_y$$

$$v_y = u_x$$

$$\nabla v = (v_x, v_y) = (-u_y, u_x)$$

Hence:
Let's define $v : \Omega \to \mathbb{R}$ by

$$v(z) = v(z_0) + \int_{z_0}^{z} (-u_y, u_x) \cdot dr$$

For this to give a well-defined function v on Ω, we need to show that the line integral of the vector field $(-u_y, u_x)$ is independent of the path from z_0 to z as long as it lies entirely in Ω.
This is **not** always true!
We need to make an assumption about the set of Ω.
Recall:
Suppose Ω is a Jordan domain.
Ω is 1-connected, (also called simply-connected), means the boundary $\partial\Omega$ consists of exactly one piece-wise smooth simple closed curve (Jordan curve).
Equivalently, the "inside" of $\partial\Omega$ of the Jordan curve lies entirely inside Ω.
Equivalently, Ω "has no holes".
Theorem:
Let Ω be a simply-connected Jordan domain. $u \in C^2(\Omega)$ is harmonic on Ω, then

$$\int_{z_0}^{z} (-u_y, u_x) \cdot dr$$

is independent of the path from z_0 to z lying entirely in Ω.
Proof:
Let's let γ_1, γ_2 be two curves in Ω from z_0 to z.
Then $\gamma + \gamma_2^{-1}$ is a closed loop based at z_0. (Simple closed curve) implies that (since Ω is simply-connected) the inside of $\gamma_1 + \gamma_2^{-1}$ lies entirely inside Ω.
Apply Green's Theorem to this curve:

$$\int_{\gamma_1} F \cdot dr - \int_{\gamma_2} F \cdot dr = \int_{\gamma_1 + \gamma_2^{-1}} F \cdot dr = \iint_D (Q_x - P_y) \, dA$$

Need to show this is zero.

$$F = (P, Q) = (-u_y, u_x)$$

$$Q_x - P_y = u_{xx} + u_{yy} = 0$$

by assumption.
Hence, the line integral is independent of path.
In general case, we decompose into a finite number of the special case.
Example:

$$u(x, y) = x^2 - y^2$$

$$F = (-u_y, u_x) = (2y, 2x) = \nabla(2xy)$$

$v = 2xy + \text{const}$ is the harmonic conjugate.

$$u + iv = \left(x^2 - y^2\right) + i\left(2xy + C\right) = z^2 + ic$$

is analytic.

$$u(x, y) = \cos(x)\cosh(y)$$

harmonic on $\mathbb{C}$.

$$\begin{aligned} F = (-u_y, u_x) &= (-\cos(x)\sinh(y), -\sin(x)\cosh(y)) \\ &= \nabla\left(-\sin(x)\sinh(x)\right) \\ v &= -\sin(x)\sin(y) + \text{const} \end{aligned}$$

In general, if Ω is a Jordan domain that is **NOT** simply-connected, then there is no guarantee that a harmonic conjugate necessarily exists.
Examples:
Puncture disc or annulus.
This is not simply connected.

$$u = \log\sqrt{x^2 + y^2}$$

This is harmonic on Ω, because there does not exist a harmonic conjugate v on Ω. On $\Omega \setminus \text{ray}$, it would have to be $\arg(z)$, which does not extend to Ω.

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Chapter 4 Complex Line integrals (also called contour intergrals)
Recall: if $F = (P, Q)$ is continous vector field on an open set Ω in $\mathbb{R}^2$ and $\gamma : [a, b] \to \mathbb{R}^2$ is a curve whose image lies inside Ω, then we define

$$\begin{aligned} \int_\gamma F \cdot dr &= \int_a^b \left(P(x(t), y(t))\frac{dx}{dt} + \left(Q(x(t), y(t))\right)\frac{dy}{dt} \right)\, dt \\ &= \int_\gamma \left(Pdx + Qdy\right) \end{aligned}$$

Let $f : \Omega \to \mathbb{C}$ be **continuous** complex-valued function on Ω.

$f = u + iv, u, v \in C^0(\Omega)$
We want to define $\int_\gamma f\ dz$
Motivation:
$z = x + iy$
"$dz = dx + idy$"
$f = u + iv.$

$$\int_\gamma f\ dz = \int_\gamma (u + iv)(dx + idy) = \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy)$$

Define:

$$\begin{aligned} \int_\gamma f\ dz &:= \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy) \\ &= \int_\gamma (u, -v) \cdot dr + i \int_\gamma (v, u) \cdot dr \end{aligned}$$

Line integrals of continuous vector fields on Ω
Aside:

$$f = u + iv \iff (u, v)$$

$$= \bar{f} \cdot dr + i \int_\gamma \left(i\bar{f}\right) \cdot dr$$

$$\bar{f} = u - iv \iff (u, -v)$$

$$i\bar{f} = v + iu \iff (v, u)$$

Circulation and flux of $\bar{f}$
Example:
$\gamma = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}$
Compute $\int_\gamma z\ dz\ f(z) = z.$

$$\gamma'(t) = (-\sin t, \cos t) = (x'(t), y'(x))$$

$$f = u + iv = x + iy$$

$$\int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy)$$

See pictures.

Note: If we went all the way around the unir circle, $a = 0, b = 2\pi$

$$\int_\gamma z\ dz = 0$$

Simplified Notation:
Let $\gamma(t) = (x(t), y(t)) \in \mathbb{R}^2 \iff \mathbb{C} \ni z(t) = x(t) + iy(t)$
Complex-valued function of $t \in [a, b]$
Define: $z'(t) = x'(t) + iy'(t)$
Make sense because $\gamma'(t) = (x'(t), y'(t))$
Proposition:

$$\int_\gamma f\ dz = \int_a^b f\left(z\left(\theta\right)\right) z'(t)\ dt$$

Proof:

$$\begin{aligned}
&= \int_a^b \left[u(x(t), y(t)) + iv(x(t), y(t))\right]\left[x'(t) + iy'(t)\right]\ dt \\
&= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy) \\
&= \int_\gamma f\ dz
\end{aligned}$$

Example:
See Pictures.
Theorem: (Analogue of fundamental theorem of calculus)
Let $f : \Omega \to \mathbb{C}$ be continuous. Suppose $\exists\ F : \Omega \to \mathbb{C}$ analytic on Ω with $F'(t) = f(z), \forall z \in \Omega$.
Then

$$\int_\gamma f\ dz = \int_\gamma F' dz = F(z_1) - F(z_0)$$

If $\gamma(a) = z_0, \gamma(b) = z_1$
Not only does this give us an easy way to compute, but it says that $\int_\gamma F' dz$ is independent of the path from z_0 to z_1 provided the path lies entirely in Ω.
Proof:
$f = u + iv, F = U + iV$

$$F' = U_x + iV_x = V_y - iU_x$$

(Cauchy-Riemann)
$f = u + iv, u = U_x = V_y, v = V_x = -U_y$

$$\gamma(b) = z_1 = x_1 + iy_1, \gamma(a) = z_0 = x_0 + iy_0$$

$$
\begin{aligned}
\int_\gamma f\ dz &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy) \\
&= \int_\gamma (U_x\ dx + U_y\ dy) + i\int_\gamma (V_x\ dx + V_y\ dy) \\
&= \int_\gamma (\nabla U)\cdot dr + i\int_\gamma (\nabla V)\cdot dr \\
&= U(x_1, y_1) - U(x_0, y_0) + i\,(V(x_1, y_1) - V(x_0, y_0)) \\
&= F(z_1) - F(z_0)
\end{aligned}
$$

Example Revisited:
Same answer for any path from 1 to i.
See pictures.
Corollary:
Suppose $\exists\ F : \Omega \to \mathbb{C}$ analytic such that $F' = f$ on Ω.
Then

$$
\int_\gamma f\ dz = 0
$$

for any **closed** curve γ lying-entirely in Ω.
Example:

$$
f(z) = \overline{z}
$$

does not appear to have an antiderivative.

$$
\begin{aligned}
\gamma_1(t) &= (t, t)\,, 0 \le t \le 1 \\
\gamma_2(t) &= \left(t, t^2\right), 0 \le t \le 1
\end{aligned}
$$

$f(z) = \overline{z}, f(z(t)) = \overline{z(t)}$
$z(t) = t + it, z'(t) = (1 + i)$

$$
\begin{aligned}
\int_{\gamma_1} \overline{z}\ dt &= \int_0^1 (t - it)(1 + i)\ dt \\
&= \int_0^1 (t - it + it + t)\,dt \\
&= t^2\Big|_0^1 = 1
\end{aligned}
$$

$f(z(t)) = x(t) - iy(t) = t - it^2, \gamma_2'(t) = (1 - i2t)$

$$
\begin{aligned}
\int_{\gamma_2} \overline{z}\ dz &= \int_0^1 \left(t - it^2\right)(1 + i2t)\ dt \\
&= \ldots
\end{aligned}
$$

We can see that they are not equal.

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One more example:

$$f(z) = \frac{1}{z}, \Omega \in \mathbb{C} \setminus \{0\}$$

Let $\gamma =$ circle centred at origin with radius $R > 0$, (counterclockwise).
Compute $\int_\gamma \frac{1}{z} dz$.
$\int_\gamma \frac{1}{z} dz = 2\pi i$
I did this in A3.
Independent of R!
Remark:
This proves (which we already knew) that does not exist a function $F : \Omega \to \mathbb{C}$ analytic on Ω with $F' = f = \frac{1}{z}$
Example 2:
Let $C(z_0; R)$ be the circle of radius R centered at z_0 such that $0 \notin \overline{D(z_0; R)}$.
then

$$\int_{C(z_0;R)} \frac{1}{z} dz = 0$$

Because we can remove a ray emanating from the origin to get a domain containing $\overline{D(z_0; R)}$ on which $\frac{1}{z}$ does have an antiderivative (a branch of logarithm).
Midterm ends here!! Up to including 4.1.2.
The M-L inequality (Very important)
Theorem:
Let Γ be a curve in $\mathbb{C}$, let f be a **continuous** complex valued function defined on Γ (usually it will be defined on a domain containing Γ).
Let $L =$ Length of γ.
Suppose that $|f(z)| \leq M, \forall z$ on Γ. (there always exists such an M because $|f(z)| = \sqrt{(u(x, y))^2 + (v(x, y))^2}$ is continuous function on a compact set $\Gamma = \gamma([a, b])$)
Extreme value theorem.
Then:

$$\left| \int_\Gamma f dz \right| \leq ML$$

Proof:

$$\mathbb{C} \ni I = \int_\Gamma f dz = |I| e^{i\omega}$$

for some phase $e^{i\omega}$ unique if $I \neq 0$.
(If $I = 0$, nothing to prove)

$$
\begin{aligned}
|I|e^{i\omega} &= \int_a^b f(z(t))z'(t)dt \\
\mathbb{R} \ni |I| &= \int_a^b e^{-i\omega} f(z(t))z'(t)dt \\
&= \int_a^b U(t)dt + i\int_a^b V(t)dt
\end{aligned}
$$

We have

$$\int_a^b U(t)dt = |I|, \int_a^b V(t)dt = 0$$

$$|I| = \int_a^b U(t)dt \leq \int_a^b |U(t)|dt$$

$$
\begin{aligned}
|U(t)| &\leq |e^{-i\omega} f(z(t))z'(t)| \\
&= |f(z(t))||z'(t)| \\
&\leq M\sqrt{(x'(t))^2 + (y'(t))^2}
\end{aligned}
$$

$$|I| \leq \int_a^b M\sqrt{(x'(t))^2 + (y'(t))^2}dt = ML$$

Example:

$$\int_\Gamma z^2 dz$$

$$L = \text{Length}\gamma = \sqrt{2}$$

$$|f(z)| = |z^2| = |z|^2 \leq \left(\sqrt{2}\right)^2 = 2 = M$$

$$|\int_\Gamma z^2 dz| \leq 2\sqrt{2}$$

by ML inequality.
We can explicitly compute this

$$= \frac{z^3}{3}\Big|_0^{1+i} = \frac{2i}{3} - \frac{2}{3}$$

$$|\int_\Gamma z^2 dz| = \sqrt{\frac{4}{9} + \frac{4}{9}} = \frac{2\sqrt{2}}{3}$$

ML inequality is not used in practice or explicit integrals.
Example: (of the type where will use ML inequality to prove theroems)
Let $f : \Omega = D(z_0; R) \setminus \{z_0\} \to \mathbb{C}$ (punctured open disc) be continuous on Ω.
Suppose $|f(z)| \leq M$ on Ω.
Suppose further that

$$I_r = \int_{C(z_0;r)} f \, dz$$

is independent of r for $0 < r < R$.
(This seemingly artificial hypothesis will arise often)
Claim: $I_r = 0, \forall r \in (0, R)$
Proof:

$$|I_r| \leq M 2\pi r, \forall r \in (0, R)$$

$$\Rightarrow |I_r| = 0$$

So $I_r = 0$
QED.

$$\int_{C(0;R)} \frac{1}{z} dz = 2\pi i$$

independent of R, but $\frac{1}{z}$ is not bounded on $D(0, R) \setminus \{0\}$.
Cauchy Integral Theorem
Theorem:
Let Ω be a domain, let $f : \Omega \to \mathbb{C}$ be **analytic** on Ω.
Let Γ be a Jordan curve (simple, closed) lying entirely in Ω whose **interior also** lies in Ω.
Then

$$\int_\Gamma f dz = 0$$

(If $f = F'$ for some analytic F, then $\int_\gamma f dz = 0$ even if γ encircles "holes")
We don't assume that $f = F'$ but we do assume f analytic.
Proof:
Let D = interior of Γ, D is a Jordan domain, $\partial D = \Gamma$.
We need to apply Green's Theorem.

$$\begin{aligned}
\int_\Gamma f dz &= \int_{\Gamma = \partial D} (u + iv)(dx + idy) \\
&= \int_{\partial D} (udx - vdy) + i \int_{\partial D} (vdx + udy) \\
&= \iint_D [(-v)_x - u_y] dA + i \iint_D (u_x - v_y) dA \\
&= 0
\end{aligned}$$

by Cauchy-Riemann equations.
Note:
The assumption that Γ does not encircle any "holes" is necessary.
$\Omega = \mathbb{C} \setminus \{0\}$ $f(z) = \frac{1}{z}$ analytic on Ω.
Picture here?
Γ is a Jordan curve.
$\int_\Gamma f dz = 2\pi i \neq 0$
Applications:

$$\int_\Gamma \cos(\sin z) dz = 0$$

$$\int_\Gamma e^{z^2} dz = 0$$

for any simple closed curve Γ in $\mathbb{C}$.
because integrands are analytic on all of $\mathbb{C}$.
Notice: We can't find explicit anti-derivatives!
Generalizations of Cauchy Integral Theorems (CIT):
Consider when the curve Γ is closed but **not** simple (it has self-intersections)
Picture here!
Suppose Γ lies inside Ω and the interior of each "lobe" lies inside Ω.
Then $\Gamma = \Gamma_1 + \Gamma_2$ when Γ_1, Γ_2 are Jordan curves whose interior lies inside Ω.

$$\int_\Gamma f dz = \int_{\Gamma_1} f dz + \int_{\Gamma_2} f dz$$

CIT -¿ 0 + 0 = 0
More generalization next time.

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Last time:
Cauchy Integral Theorem (CIT)
Let $f : \Omega \to \mathbb{C}$ be analytic on Ω. Let Γ be a Jordan curve (simple closed) lying in Ω and whose interior lies in Ω.
Then

$$\int_\Gamma f dz = 0$$

Generalization of the Cauchy Integral Theorem
Last time we argued that the CIT still holds for closed curves with a finite number of self-intersection. (non-simple) as long as all the "interiors" of Γ lies inside Ω.
To make a "precise" statement of this, requires the language of **homotopy of path**.
We'll give a less precise statement.
Strong Cauchy Integral Theorem (SCIT)

Let Ω be a domain, $f : \Omega \to \mathbb{C}$ analytic on Ω.
Let Γ be a closed curve lying in Ω (Not necessary simple) which can be shrunk to a point continuously without leaving Ω.
Then $\int_\Gamma f dz = 0$.
Example:
See pictures.
Important Special Case of SCIT:
Suppose Ω is simply-connected, (it has "no holes").
Ω is simply-connected $\iff$ any closed curve Γ in Ω can be shrunk to a point without leaving Ω.
Aside:
If Ω is a Jordan domain, then Ω is simply-connected if and only if Ω is 1-connected. ($\partial\Omega$ consists of single Jordan curve)
But the definition of simple-connectedness for a domain Ω does not require Ω to be a Jordan domain. (It doesn't even have to be bounded)
Corollary of SCIT:
If Ω is a simply-connected domain, and $f : \Omega \to \mathbb{C}$ analytic on Ω.
Then $\int_\Gamma f dz = 0$ for any closed curve Γ lying in Ω.
Another Corollary of SCIT:
Let Ω be a domain, let $f : \Omega \to \mathbb{C}$ be analytic on Ω.
Let Γ_1, Γ_2 be two (piecewise smooth) curves in Ω from z_0 to z_1, such that all points "between the two curves" lie in Ω.
Then

$$\int_{\Gamma_1} f dz = \int_{\Gamma_2} f dz$$

"independence of path"
Proof:
Apply SCIT to $\Gamma_1 + \Gamma_2^{-1}$.
Corollary
Let Ω be a simply-connected domain. Let $f : \Omega \to \mathbb{C}$ be analytic on Ω. Let $z_0, z \in \Omega$. Define

$$F : \Omega \to \mathbb{C}$$

by

$$F(z) = \int_{z_0}^{z} f(w)dw$$

for any curve in Ω from z_0 to z.
Then F is well-defined and continous on Ω.
Proof:
Well-defined is immediate from previous Corollary. We need to show $F : \Omega \to \mathbb{C}$ is continuous on Ω.
Let $z_1 \in \Omega$, we need to show that F is continous at z_1.
So we need to show that

$$\lim_{z \to z_1} F(z) = F(z_1) \iff \lim_{z \to z_1} (F(z) - F(z_1)) = 0$$

$$\begin{aligned}|F(z)-F(z_1)| &= |\int_{z_0}^{z} f(w)dw - \int_{z_0}^{z} f(w)dw| \\ &= |\int_{z_0}^{z} f(w)dw + \int_{z_1}^{z_0} f(w)dw| \\ &= |\int_{z_1}^{z} f(w)dw| \\ &\leq \sup_{w \text{ on this curve } z_1 \to z} (\text{length of this curve})\end{aligned}$$

f is continous on Ω any curve is a continuous set, so f is boudned on the curve. So by Squeeze Theorem,

$$\lim_{z \to z_1} F(z) = F(z_1)$$

So F is continous on Ω.
In fact, it is true that F is **analytic** and $F' = f$. We will prove a more general result:
Theorem:
Let $f : \mathbb{C} \to \mathbb{C}$ be continuous on Ω and suppose $F(z) = \int_{z_0}^{z} f(w)dw$ is **independent of path** for any polygonal (Piecewise linear) paths from z_0 to z in Ω.
Then F is analytic on Ω and $F' = f$.
Note:
If f is analytic and Ω is simply-connected, then

$$\int_{z_0}^{z} f(w)dw$$

is independent of path for any path in Ω from z_0 to z.
Before we prove this, let's state an amazing corollary.
Corollary:
If $f : \Omega \to \mathbb{C}$ is analytic on a simply-connected domain Ω.
Then $f = F'$ where $F(z) = \int_{z_0}^{z} f(w)dw$ for any path from z_0 to z in Ω.
(So on a simply-connected domain, any analytic function is the complex derivative of analytic functions)

$$\frac{d}{dz}\int_{z_0}^{z} f(w)dw = f(z)$$

"Fundamental Theorem of Calculus"
Proof:
Let $z_1 \in \Omega$.
Need to show $\lim_{z \to z_1}$ of above is $f(z_1)$
Pictures.

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In pictures.

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Cauchy Integral Formula
Let $f : \Omega \to \mathbb{C}$ be analytic on a domain Ω.
Let Γ be a Jordan curve in $\mathbb{R}$ with $\text{int}(\Gamma) \subseteq \Omega$.
Let $z_0 \in \text{int}\,(\Gamma)$

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz$$

Characterization of complex differentiability (analyticity) using real partial derivative.
Aside:
Let $f(z) = u(x, y) + iv(x, y), u, v \in C^1\,(\Omega)$
If f is analytic on Ω, then

$$\begin{aligned}
\frac{\partial f}{\partial z} &= u_x + iv_x = v_y - iv_x \\
&= \frac{1}{2}\,(u_x + iv_x) + \frac{1}{2}\,(v_y - iu_y) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u + \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(iv) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv)
\end{aligned}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

when f is analytic.
Define: (Just notation)

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

This is a complex-valued first order linear differential operator.
Suppose as before $f = u + iv, u, v \in C^1$
Compute

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)\\ &= \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y)\\ &= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)\end{aligned}$$

So $\frac{\partial f}{\partial \overline{z}} = 0$
Cauchy-Riemann equations are satisfied.
f is analytic.
Summary:
$f = u + iv, u, v \in C^1(\Omega)$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z} =$$

See pictures.
Next, we prove a **complex version** of "differentiation under integral sign".
Lemmas:
Let Ω be a domain. Let Γ be a **curve** (not necessarily simple or closed.)
Such that $\Gamma \cap \Omega = \emptyset$. Let $F(z,\zeta)$ be continous $\forall z \in \Gamma, \zeta \in \Omega$ and analytic in $\zeta \in \Omega$
So $\frac{\partial F}{\partial \zeta}(z,\zeta)$ is continuous $\forall z \in \Gamma, \zeta \in \Omega$.
Then $\int_\Gamma F(z,\zeta)$ is analytic $\forall \zeta \in \Omega$.
and

$$\frac{d}{d\zeta}\int_\Gamma F(z,\zeta)dz = \int_\Gamma \frac{\partial F}{\partial \zeta}(z,\zeta)dz$$

Proof:
Let $z = z(s)$ on $\Gamma, a \leq s \leq b$.
See pictures.
...
Let's apply this to Cauchy Integral Formula.
Let $f : \Omega \to \mathbb{C}$ be analytic on a domain Ω.
Let Γ be a Jordan curve in Ω such that $\operatorname{int}(\Gamma) \subseteq \Omega$ and let $\zeta \in \operatorname{int}(\Gamma)$
Then

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\zeta}dz$$

Integrand is $\frac{f(z)}{z-\zeta}$ is continous $\forall z \in \Gamma$ and $\zeta \in \operatorname{int}(\Gamma)$ and it is analytic in $\zeta \in \operatorname{int}(\Gamma)$ with

$$\frac{\partial}{\partial \zeta}\left(\frac{f(z)}{z-\zeta}\right) = \frac{f(z)}{(z-\zeta)^2}$$

Continuous for all $z \in \Gamma, \zeta \in \operatorname{int}(\Gamma)$
We can apply the lemma.
Ω in lemma $\Leftrightarrow \int(\Gamma)$ here.
Lemma says $f(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-\zeta} dz$ is analytic in ζ and

$$\frac{\partial}{\partial \zeta}\left[\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-\zeta} dz\right] = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-\zeta)^2} dz$$

$$f'(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-\zeta)^2} dz$$

First Generated Cauchy Integral Formula.
Now, do it again.

$$\begin{aligned}
\frac{\partial}{\partial \zeta} f'(\zeta) &= f''(\zeta) \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{\partial}{\partial \zeta}\left(\frac{f(z)}{(z-\zeta)^2}\right) dz \\
&= \frac{2}{2\pi i} \int_\Gamma \frac{f(z)}{(z-\zeta)^3} dz
\end{aligned}$$

Keep repeating:
We've proved f is infinitely (complex) differentiable at all $\zeta \in \operatorname{int}(\Gamma)$, and

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-\zeta)^{n+1}} dz$$

Notice: $n = 0$ gives original CIF.
What we have shown is:
Suppose f is complex-valued function on some domain Ω, and its analytic at $\zeta \in \Omega$.
Then, it's analytic on $D(\zeta, \epsilon)$ for some $\epsilon > 0$.
Apply GCIF to $\Gamma = C(\zeta, \frac{\epsilon}{2})$
$\Rightarrow f$ is infinitely complex differentiable at ζ.
"Applications"

$$\int_\Gamma \frac{f(z)}{(z-\zeta)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(\zeta)$$

allows us to compute certain integrals of this form for f analytic.
Exercise:
$\zeta = 0, n = 5, f(z) = \sin(z)$ analytic everywhere.

$$
\begin{aligned}
\int_{|z|=1} \frac{\sin|z|}{z^6} dz &= \int_\Gamma \frac{\sin(z)}{(z-0)^{5+1}} dz \\
&= \frac{2\pi i}{5!} f^{(5)}(0) \\
&= \frac{2\pi i}{120} \cos(0) \\
&= \frac{\pi i}{60}
\end{aligned}
$$

$$
\int_\Gamma \frac{\tan(\frac{z}{2})}{\left(z-\frac{\pi}{2}\right)^2} dz = \int_\Gamma \frac{f(z)}{(z-\zeta)^{1+1}} dz
$$

See pictures.

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Generalized Cauchy Integral Formula
Let $f : \Omega \to \mathbb{C}$ be analytic on a domain Ω. Then f is infinitely differentiable on Ω.
Let Γ be a Jordan curve in Ω with $\operatorname{int}(\Gamma) \subseteq \Omega$.
Let $z_0 \in \operatorname{int}(\Gamma)$
Then

$$
f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} dz
$$

Can be used to evaluate certain integrals.
Example:

$$
\int_\Gamma \frac{\cosh(z)}{z^4} dz
$$

$x = \pm 2, y = \pm 2$. A square.
$f(z) = \cosh(z)$, analytic everywhere.
$\zeta = 0, n = 3$

$$
\int_\Gamma \frac{\cosh(z)}{z^4} dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{6} \sinh(0) = 0
$$

We notice that $\frac{\cosh(z)}{z^4}$ is not analytic in $\operatorname{int}(\Gamma)$.
So $\int_\Gamma h(z)dz = 0 \not\Rightarrow$ h analytic in $\operatorname{int}(\Gamma)$
The converse is the CIT.
Corollary:
Let $f = u + iv$ be analytic on a domain Ω.
Then u, v are **harmonic** on Ω.

Proof:
We already showed this is true provided $u, v \in C^2(\Omega)$.
But now we know this is **always true** because f''' exists so f'' continuous on Ω.
$f = u + iv$
$f' = u_x + iv_x = v_y - iu_y$
$f'' = u_{xx} + iv_{yy} = v_{yx} - iu_{yx} = v_{xy} - iu_{xy} = -u_{yy} - iv_{yx}$
continous on Ω.
$u_{xx}, u_{xy}, u_{yx}, u_{yy}, v_{xx}, v_{xy}, v_{yx}, v_{yy} \in C^2(\Omega)$
$\Rightarrow u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$

$$\begin{aligned} &u_{xx} + u_{yy} \\ =&(u_x)_x + u(y)_y \\ =&(v_y)_x - (v_x)_y = 0 \end{aligned}$$

So if $f = u + iv$ is analytic on Ω, then $u, v \in C^\infty(\Omega)$ and harmonic on Ω.
Circumferential Mean Value Property
Let $f : \Omega \to \mathbb{C}$ be analytic on a domain Ω. Let $\overline{D(z_0), R} \subseteq \Omega$.
Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \left(f(z_0) + Re^{i\theta}\right) d\theta$$

Average of f over $C(z_0, R)$ equals value at centre.
Proof:

$$f(z) = u(z) + iv(z)$$

Harmonic on Ω

$$\begin{aligned} f(z_0) = u(z_0) + iv(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\theta}) d\theta + i \int_0^{2pi} v(z_0 + Re^{i\theta}) dQ \\ = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta \end{aligned}$$

Remarks:
On A4, you give a different proof using CIF.
Solid Mean Value Property
$f : \Omega \to \mathbb{C}$ analytic on a domain Ω.
$\overline{D(z_0, R)} \subseteq \Omega$.
Then,

$$f(z_0) = \frac{1}{\pi R^2} \iint_{D(z_0, R)} f dA$$

Proof:
Apply Solid MVP to u, v (harmonic)

$$
\begin{aligned}
f(z_0) &= u(z_0) + iv(z_0) \\
&= \frac{1}{\pi R^2}\iint_D u dA + i\frac{1}{\pi R^2}\iint_D v dA \\
&= \frac{1}{\pi R^2}\iint_D f dA
\end{aligned}
$$

Example:

$$\int_0^{2\pi} \log(2+\epsilon e^{i\theta})d\theta = 2\pi \log 2 > 0$$

Using CMVP

$$\int_{|z-2|=\epsilon} \log(z)dz = 0$$

Using CIT
Because log is analytic on an open set containing $\overline{D(2,\epsilon)}$.
Let's see explicitly that these are not the same.

$$z(\theta) = 2+\epsilon e^{i\theta}, 0 \le \theta \le 2\pi$$

Parametrizes $C(2;\epsilon)$.

$$
\begin{aligned}
\int_{C(2;\epsilon)} \log(z)dz &= \int_0^{2\pi} \log(z(\theta))z'(\theta)d\theta \\
&= \int_0^{2\pi} \log(2+\epsilon e^{i\theta})\epsilon i e^{i\theta} d\theta
\end{aligned}
$$

Example:

$$\iint_{D(0;1)} \cos(z)dA$$

Using the SMVP:

$$\pi(1)^2\cos(0) = \pi$$

Maximum / Minimum Principles
Recall: The strong / weak maximum / minimum principles for **harmonic functions**.
SMP: Let Ω be a domain. A **non-constant** harmonic function on Ω does **not** attain a global maximum nor a global minimum on Ω.

WMP: Let Ω be a **bounded** domain. Suppose $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ and **harmonic** in Ω.
If u is non-constant, then u attains its global maximum and global minimum on $\partial\Omega$ only.
These results do not directly generalize to analytic functions (which are complex-valued!) because $\mathbb{C}$ is not natually ordered.
It does not make sense $z, w \in \mathbb{C}, z \leq w$.
Given $z, w \in \mathbb{C}$, $|z|, |w|$ are **real**.
So $|z| \leq |w|$ makes sense.
Definition:
Let Ω be a domain. Let $f : \Omega \to \mathbb{C}$ be a function on Ω.
We say f attains a **maximum modulus** on Ω if $\exists z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|, \forall z \in \Omega$.
(i. e if the real-valued function $z \mapsto |f(z)|$ has a global maximum at z_0)
Strong Maximimum Modulus Principle
Ω a domain.
$f : \Omega \to \mathbb{C}$ analytic on Ω, and **non-constant**. Then $|f(z)|$ does not attain a global max on Ω.
Proof:
If $f(\Omega) = \{w \in \mathbb{C}, w = f(z) \text{ for some } z \in \Omega\}$ is unbounded, then the result is clear.
So we can assume $f(\Omega)$ is bounded.
Suppose f does attain a maximum modulus on Ω.
So $\exists z_0 \in \Omega$ such that

$$0 \leq |f(z)| \leq |f(z_0)|, \forall z \in \Omega$$

Note:
$f(z_0) \neq 0$ (if so, $f(z) = 0, \forall t \in \Omega$, but f is nonconstant)
So $|f(z_0)| > 0$, $f(z_0) \neq 0$.
Define $g(z) = f(z) + c \cdot f(z_0)$ where c is **real** and $c \geq 1$.
Triangle Inequality:

$$\begin{aligned}|g(z)| &\geq c \cdot |f(z_0)| - |f(z)| \\ &= (c-1)|f(z_0)| + |f(z_0)| - |f(z)| \geq (c-1)|f(z_0)| > 0.\end{aligned}$$

So $g(z) \neq 0, \forall z \in \Omega$.
g is analytic on Ω. $g(z) \neq 0, \forall z \in \Omega$.
We will show (next time) that g also has a max modulus at z_0. Then we will get a contradiction.

21 March 2nd

Strong Maximum Modulus Principle
Let $\Omega \subseteq \mathbb{C}$ be a domain.
Let $f : \Omega \to \mathbb{C}$ be analytic and **non-constant**.

Then f does not attain a global maximum modulus on Ω.
(i.e There does not exist $z_0 \in \Omega$ with $|f(z)| \leq |f(z_0)|, \forall z \in \Omega$)
Proof:
Last time:
It is clear if $f(\Omega)$ is unbounded. Assume $f(\Omega)$ is bounded and such a $z_0 \in \Omega$ exists.

$$|f(z)| \leq |f(z_0)|, \forall z \in \Omega$$

So $f(z_0) \neq 0$.
Let $g(z) = f(z) + cf(z_0), c > 1$

$$|g(z)| \geq c|f(z_0)| - |f(z)| > 0$$

Claim: g has a maximum modulus at z_0.
$\forall z \in \Omega$

$$|g(z)| \leq |f(z)| + c|f(z_0)| \leq |f(z_0)| + c|f(z_0)| = (1+c)|f(z_0)| = |g(z_0)|$$

Notice: g is analytic on Ω, $g(z) \neq 0, \forall z \in \Omega$, and $|g(z)| \leq |g(z_0)|, \forall z \in \Omega$.
We need to get a contradiction.
We can define a branch of the logarithm such that $g(\Omega)$ lies in its domain.
Let $h(z) = \log(g(z))$, this is analytic on Ω.

$$\log|g(z)| + i\arg(g(z))$$

The real and imaginary parts are harmonic on its domain.
$\log|g(z)|$ is harmonic on Ω and attains a global maximum at z_0. (Because $\log(0,\infty) \to \mathbb{R}$ is strictly increasing and $|g(z)|$ has a global max at z_0)
By Strong Maximum Principle for harmonic functions, applied to the function $\log|g(z)|$, we conclude that $|g(z)|$ is constant.
On earlier assignment, you proved if g is analytic on Ω and $|g(z)|$ is constant, then g is constant.
Contradiction! So z_0 does not exist.
(Need to take c sufficiently large so that $g(\Omega)$ lies on one side of a line through origin. We can do this because $g(\Omega)$ is bounded)
Weak Maximum Modulus Principle
Let Ω be a **bounded domain** and $f : \Omega \to \mathbb{C}$ continuous on $\overline{\Omega}$ and analytic on Ω.
Then f attains its global maximum modulus on $\overline{\Omega}$ only at points in $\partial\Omega$.
(By Extreme Value Theorem and Strong Maximum Modulus Principle)
Minimum Modulus Principle
Let Ω be a domain. Let $f : \Omega \to \mathbb{C}$ be **non-constant** and analytic on Ω, with $f(z) \neq 0, \forall z \in \Omega$.
Then f does not attain a global minimum modulus anywhere on Ω.
If there exist $z_0 \in \Omega$ with $f(z_0) = 0$, then f does attain a global minimum modulus at z_0.
Proof:
Since $f(z) \neq 0, \forall z \in \Omega$.

$$h(z) = \frac{1}{f(z)}$$

is analytic on Ω.

$$|h(z)| = \frac{1}{|f(z)|}$$

has a global maximum
Then, $|f(z)|$ has global maximum which it doesn't by SMMP.
Proposition:
Let f, g be analytic on Ω (domain).
Let Γ be a Jordan curve in Ω with $\text{int}(\Gamma) \subseteq \Omega$.
If $f(z) = g(z), \forall z \in \Gamma$, then $f(z) = g(z), \forall z \in \text{int}(\Gamma)$
Proof:
$h(z) = f(z) - g(z) = 0$ on $\Gamma = \partial(\text{int}\Gamma)$ $|h(z)| = 0, \forall t \in \text{int}\Gamma$ $f(z) = g(z)$ on $\text{int}\Gamma$.
Recall Liouville's Theorem for harmonic functions:
Let $u : \mathbb{R}^2 \to \mathbb{R}$ be **harmonic** on the entire plane (entire harmonic).
If u is non-constant, u is **unbounded**.
Theorem: Liouville's Theorem for entire analytic functions.
Theorem: Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on entire plane (entire analytic)
If f is non-constant, it implies that f is **unbounded**.
Equivalence: If f is bounded, it implies that f is constant.
Proof: Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic and $|f(z)| \leq M \forall z$. $|u(z)| \leq M, |v(z)| \leq M, \forall z$
Using Liouville for entire harmonic.
u, v constant. So $f = u + iv$ is constant.
Fundamental Theorem of Algebra
Theorem: Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients. $a_i \in \mathbb{C}, i = 0, \ldots, n, a_n \neq 0$
Then there exist $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.
(i.e every non-constant polynomial over $\mathbb{C}$ has at least root in $\mathbb{C}$ hence at most n roots)
Proof:
First, we claim that given any positive real number, $M > 0$, $\exists R > 0$ such that $|p(z)| \geq M, \forall |z| \geq R$
("p" goes to infinity as $z \to \infty$)
Proof of claim:
Choose $k > 0$ such that

$$\left|\frac{a_{n-k}}{z^k}\right| \leq \frac{|a_n|}{2n}, \forall |z| \geq R, k = 1, \ldots, n$$

(Take $R \geq \max_{k=1} \left[\left(\frac{2n|a_{n-k}|}{|a_n|}\right)^{\frac{1}{k}}\right]$, $a_n \neq 0$)
Then $|z| \geq R \Rightarrow |z|^k \geq \frac{2n|a_{n-k}|}{|a_n|}$

By Triangle inequality,

$$|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}| \leq \frac{n|a_n|}{2n} = \frac{|a_n|}{2}$$

for $|z| \geq R$.

$$|\frac{a_{n-k}}{z^k}| \leq \frac{|a_n|}{2n}$$

$k = 1, \ldots, n$

$$P(z) = z^n(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n})$$
$$|P(z)| = |z|^n \left| a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) \right|$$
$$\geq |z|^n \left(|a_n| = \frac{|a_n|}{2} = |z|^n \frac{|a_n|}{2} \right)$$

for $|z| \geq R$.
Using triangle inequality.
Let $R \geq \left(\frac{2M}{|a_n|}\right)^{\frac{1}{n}}$
$|z| \geq R \Rightarrow |z|^n \geq R^n = \frac{2M}{|a_n|}$
...
See pictures.

22 March 4th

Liouville's Theorem for entire analytic functions
Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on the entire plane $\mathbb{C}$ if f is bounded $\Rightarrow f$ is constant.
Equivalently if f is non-constant $\Rightarrow f$ is unbounded.
Example:
Let $f : \mathbb{C} \to \mathbb{C}$ be entire analytic and suppose that $|f(z)| \leq |e^z|, \forall z \in \mathbb{C}$.
Then $f(z) = c \cdot e^z$ for some $c \in \mathbb{C}$ with $|c| \leq 1$.
Proof:
$h(z) = \frac{f(z)}{e^z}$ is entire analytic. By hypothesis,

$$|h(z)| = \frac{|f(z)|}{|e^z|} \leq 1$$

By Liouville, $h(z) = c, |c| = |h(z)| \leq 1$.
You may have heard that a polynomial grows slower than an exponential in real analysis.
Suppose $f(z)$ is a polynomial, let $z = x$ be real and very negative, $|f(x)|$ is large, but $|e^x|$ is small.

$$|f(x)| \not\leq |e^x|$$

for x large negative.
The Cauchy Inequalities for $f^{(n)}(z_0)$
Let $f : \Omega \to \mathbb{C}$ be analytic on Ω. Let $\overline{D(z_0; R)} \subseteq \Omega$.
Let $M(z_0; R) = \max_{C(z_0;R)} |f(z)|$.
Then,

$$|f^{(n)}(z_0)| \leq \frac{n!M(z_0; R)}{R^n}$$

Proof:
By GCIF,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C(z_0;R)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

By ML inequality,

$$\begin{aligned} |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \frac{M(z_0; R)}{R^{n+1}} 2\pi R \\ &= n! \frac{M(z_0; R)}{R^n} \end{aligned}$$

Remark: Normally, bounds on derivatives of a function are used to obtain bounds on the function itself.
This result shows that for **analytic** functions, we can also do the opposite. Bounds on the function yield bounds on all of its derivatives.
Corollary of Cauchy inequality
A different proof of Liouville Theorem.
Suppose f is entire analytic and bounded.

$$|f(z)| \leq M, \forall z \in \mathbb{C}$$

$$M(z_0; R) \leq M, \forall z_0 \in \mathbb{C}, \forall R > 0$$

By Cauchy Inequality for $n = 1$,

$$|f'(z_0)| \leq \frac{M}{R}$$

for any $R > 0$, $\Rightarrow f'(z_0) = 0, \forall z_0 \Rightarrow f$ const.
Morera's Theorem
Let $R : \Omega \to \mathbb{C}$ be **continuous** on a domain Ω, and suppose that $\int_\Gamma f(z)dz = 0$ for any Jordan curve Γ lying in Ω.
Then f is analytic on Ω.
(TEXT INCORRECTLY SAYS WE NEED $\text{int}(\Gamma) \subseteq \Omega$)
Remark: If $\text{int}(\Gamma) \subseteq \Omega$ and f is analytic, then $\int_\Gamma f(z)dz = 0$ by CIT.
So Morera's Theorem is a kind of converse to CIT.
Remark:
Morera's Theorem gives a **sufficient** but **not necessary** condition for analyticity.

Example:
$f(z) = \frac{1}{z}$ is analytic on $\Omega = \mathbb{C} \setminus \{0\}$, but $\int_\Gamma f(z)dz \neq 0$ if Γ encircles origin.
Proof:

$$\int_\Gamma f(z)dz = 0, \forall \text{Jordan curves} \Gamma \text{ in } \Omega$$

$$\iff \int_{z_0}^{z} f(z)dz \text{ is path independent for any path from } z_0 \text{ to } z \text{ lying in } \Omega$$

By earlier theorem, if $\int_{z_0}^{z} f(\zeta)d\zeta$ is path-independent for all polygonal (piecewise-linear) paths in Ω, then

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

is analytic on Ω and $F' = f$
Now we use the fact that:
Since F is analytic, F is infinitely differentiable and all its derivatives are analytic on Ω.
(This was a corollary of GCIF.)
So $F' = f$ is differentiable on Ω.
$F'' = f'$ is differentiable, hence continous on Ω.
So f is analytic on Ω.
From the proof, it is clear that the hypotheses can be weakened to

$$\int_\Gamma f(z)dz = 0$$

for any closed polygonal (piecewise-linear) curve in Ω.
It then follows that we can further restrict to triangles.
Application of Morera's Theorem
Removable Singularities.
Let $D = D(z_0; R)$ be an open disc centred at $z_0 \in \mathbb{C}$.
Suppose $f : D \to \mathbb{C}$ is **continuous** on D and **analytic** on $D \setminus \{z_0\}$ (punctured disc).
Then f is also analytic at z_0.
Proof:
Since f is continous on D, $\int_\Gamma f(z)dz$ exists for any Jordan curve Γ in D, There are three cases.

1. $z_0 \notin \text{int}(\Gamma), z_0 \not in \Gamma$
2. $z_0 \in \text{int}(\Gamma)$
3. $z_0 \in \Gamma$

So if we can show $\int_\Gamma f(z)dz = 0$ in all three cases, then Morera's Theorem tells us f is analytic on D.
If $z_0 \notin \Gamma$, $z_0 \notin \text{int}(\Gamma)$.

$$\int_\Gamma f(z)dz = 0$$

by CIT.
Because f is analytic on open set containing Γ, $\text{int}\,(\Gamma)$.
Case 2:
$z_0 \in \text{int}(\Gamma)$. By earlier results,

$$\int_\Gamma f(z)dz = \int_{C(z_0;\epsilon)} f(z)dz$$

provided ϵ is sufficiently small so that $\overline{D(z_0;\epsilon)} \subseteq \text{int}(\Gamma)$.
By ML,

$$\left|\int_{C(z_0;\epsilon)} f(z)dz\right| \leq M2\pi\epsilon$$

M : max of f on $\overline{D(z_0;\frac{R}{2})}$.
Let $\epsilon < \frac{R}{2}$.
For all ϵ sufficiently small,

$$\int_\Gamma f(z)dz = \int_{C(z_0;\epsilon)} f(z)dz = 0$$

Case 3: $z_0 \in \Gamma$.
Let $\tilde{\Gamma}$ differ from Γ, only "near z_0" such that $\tilde{\Gamma}$ is of case 1 or 2.

$$\int_{\tilde{\Gamma}} f(z)dz = 0$$

$$\int_\Gamma f(z)dz - \int_{\tilde{\Gamma}} dz = \int_B f(z)dz$$

(By continuity and ML inequality).
So $\int_\Gamma f(z)dz = 0$ for any Jordan curve Γ in D.
So f is analytic on D.
Later in this course, we will give another proof of "Removable singularities" using power series.
Schwarz Lemma (Shows again how rigid analytic functions are)
Let $D = D(0,1)$ be the open unit disc centered at origin.
Let $f : D \to \mathbb{C}$ be **analytic** on D and such that

1. $f(0) = 0$
2. $|f(z)| \leq 1, \forall z \in D$.

   (2) says f maps open unit disc into the closed unit disc.

Then, we must have

$$|f(z)| \leq |z|, \forall z \in D$$

and

$$|f'(0)| = 1$$

.
Moreover, if equality holds in (A) for some $z \in D$, or equality holds for B. (i. e) if $\exists z \in D_{z \neq 0}$ such that $|f(z)| = |z|$ or if $|f'(0)| = 1$.
Then $f(z) = e^{i\theta} z$ for some constant $e^{i\phi}$.
Hence $|f(z)| = |z|, \forall z \in D$ and $|f'(z)| = 1, \forall z \in D$.
Rotation about origin by angle ϕ.

23 May 6th

Recall: Removable Singularities
Let $D : D(z_0; R)$ if $f : D \to \mathbb{C}$ is continuous on D and analytic in $D \setminus \{z_0\}$, then it is analytic on D.
Schwarz Lemma
Let $D = D(0, 1)$ be the unit open disc.
Let $f : D \to \mathbb{C}$ be analytic on D such that

1. $f(0) = 0$
2. $|f(z)| \leq 1, \forall z \in D$

(Maps open unit disc into the closed unit disc).
Then

1. $|f(z)| \leq |z|, \forall z \in D$
2. $|f'(0)| \leq 1$

Moreover, if equality holds in (A) for some $z \neq 0$ in D or if equality holds in (B). Then

$$f(z) = e^{i\theta} z$$

for some constant $e^{i\theta}$
Hence, $|f'(z)| = 1, \forall z \in D, |f(z)| = |z|, \forall z \in D$.
Proof:
Define $g : D \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{z}$$

for $z \neq 0$.
$g(0) = f'(0)$
By construction, g is analytic on $D \setminus \{0\}$ and continuous on D.
Hence, by removable singularities, g is analytic on D.
Let $0 < r < 1$, let $\overline{D}_r = \overline{D(0, r)}$.

By the weak maximum modulus principle, $\exists z_r \in \partial\overline{D}_r = \{z \in \mathbb{C}, |z| = r\}$ such that $|g(z)| \leq |g(z_r)|, \forall z \in \overline{D}_r$.
But

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}$$

We get $|g(z)| \leq \frac{1}{r}, \forall z \in \overline{D}_r$ $|z_r| = r, |f(z)| \leq 1, \forall z \in D$
$\Rightarrow |g(z)| \leq 1, \forall z \in D$

$$|g(0)| = |f'(0)| \leq 1$$

by (B)

$$z \neq 0, |g(z)| = \frac{f(z)}{|z|} \leq 1, \Rightarrow |f(z)| \leq |z|, \forall z \in D$$

Now, suppose equality holds in (A), for some $z \neq 0$ or equality holds in (B).
Then $|g(z)| = 1$ for some $z \in D$.
But $|g(z)| \leq 1, \forall z$, so g attains a global maximum on D.
By Strong Maximimum Modulus Principle, (since g analytic on D), we must have $g(z) = c$ **constant**.

$$|c| = |g(z)| = |g(z_0)| = 1$$

So

$$c = e^{i\theta}$$

Chapter 5: Complex Series, and Power Series over $\mathbb{C}$ (and relation to analyticity)
Review **real series**
Let $a_k \in \mathbb{R}, \forall k \geq 0$.
We say $\sum_{k=0}^{\infty} a_k$ converges iff $\lim_{N\to\infty}\left(\sum_{k=0}^{N} a_k\right)$ exists.
Since a sequence of real numbers has a limit iff it is a Cauchy Sequence, we get $\sum_{k=0}^{\infty} a_k$ converges iff $\forall \epsilon > 0, \exists N \geq 0$ such that $N \leq n \leq m$.
Then $|a_n + \cdots + a_m| < \epsilon, |S_m - S_{n-1}| < \epsilon$.
Complex Series
Let $c_k = a_k + ib_k \in \mathbb{C}, \forall k \geq 0$.
We say $\sum_{k=0}^{\infty} c_k$ converges iff $\lim_{N\to\infty}\left(\sum_{k=0}^{N} c_k\right)$ exists.

$$\iff \lim_{N\to\infty}\left(\sum_{k=0}^{N} a_k\right) + i\left(\sum_{k=0}^{N} b_k\right)$$

exists.
If $(a_k, b_k) \in \mathbb{R}^2$ is a sequence, then $(a_k, b_k) \to_{k\to\infty} (a, b) \in \mathbb{R}^2$.
Iff $a_k \to_{k\to\infty} a$ and $b_k \to_{k\to\infty} b$
Because,

$$||(a_k, b_k) - (a, b)||^2 = |a_k - a|^2 + |b_k - b|^2$$

Hence, $\sum_{k=0}^{\infty} c_k$ converges iff both $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge.

And

$$\sum_{k=0}^{\infty}(a_k + ib_k) = \left(\sum_{k=0}^{\infty} a_k\right) + i\left(\sum_{k=0}^{\infty} b_k\right)$$

Absolute COnvergence
Let $\sum_{k=0}^{\infty} c_k$ be a complex series.

$$\lim_{N\to\infty}\left(\sum_{k=0}^{N} c_k\right)$$

We say the sequence **converges absolutely** iff $\sum_{k=0}^{\infty} |c_k|$ converges.
$\lim_{N\to\infty} \sum_{k=0}^{N} |c_k|$ exists.
Proof:
If $\sum_{k=0}^{\infty} c_k$ converges absolutely, then it converges.
Proof:
Let $\epsilon > 0, \exists N \geq 0$ such that $N \leq n \leq m$ then

$$|c_n| + \cdots + |c_m| \leq \epsilon$$

Because $\sum_{k=0}^{\infty} |c_k|$ converges.
By Triangle inequality,

$$|c_n + \cdots + c_m| \leq |c_n| + \cdots + |c_m| < \epsilon, \forall m \geq n \leq N$$

Hence $\sum_{k=0}^{\infty} c_k$ converges.
There exist series that converge, but **do not** converge absolutely.
Example:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$$

Proposition:
Suppose $\sum_{k=0}^{\infty} c_k$ converges, then

1. $\lim_{k\to\infty} c_k = 0$ (Terms go to zero)
2. $\exists M > 0$ such that $|c_k| \leq M, \forall k$ (Terms are bounded).

Both necessary for convergence but not sufficient.
Proof:
Let $\epsilon > 0, \exists N \geq 0$ such that $N \leq n \leq m$.

$$|c_n + \cdots + c_m| < \epsilon$$

Take $m = n$.
$|c_n| < \epsilon$
$\Rightarrow \lim_{k\to\infty} c_k = 0)$
$1 \to 2$ is trivial.

Finally, **Comparison Test**
Let $0 \leq a_k \leq b_k, \forall k \geq 0$.
If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.
If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.
(Exercise)
Power Series
Start with the **most** important example,
Let $z \in \mathbb{C}$, consider $\sum_{k=0}^{\infty} z^k, (c_k = z^k)$ (Geometric Series)

$$S_N = 1 + z + z^2 + \cdots + z^N$$
$$zS_n = z + z^2 + \cdots + z^N + z^{N+1}$$
$$S_n - zS_n = 1 - z^{N+1}$$

Suppose $z \neq 1$, $S_n = \frac{1-z^{N+1}}{1-z}$
Suppose $|z| < 1, \lim_{N\to\infty} z^{N+1} = 0$.
Because $|z^{N+1} - 0| = |z^{N+1}| = |z|^{N+1} \to 0$ as $N \to \infty$.
Hence, if $|z| < 1$, then $\sum_{k=0}^{\infty} z^k$ converges to $\frac{1}{1-z}$.
If $|z| \geq 1$, then
$$|c_k| = |z^k| = |z|^k \geq 1, \forall k$$
So $c_k \not\to 0$.
Hence, $\sum_{k=0}^{\infty} c_k$ does not converge.
Summary:

$$\sum_{k=0}^{\infty} z^k$$

converges iff $|z| < 1$ and if so it converges to $\frac{1}{1-z}$.
General Power Series
Let $z_0 \in \mathbb{C}$ be fixed. A power series centred at z_0 is a complex series of the form $\sum_{k=0}^{\infty} c_k(z - z_0)^k$, $c_k \in \mathbb{C}, \forall k$, $z \in \mathbb{C}$. k non-ngeative integer.
Given a power series centred at z_0, consider z_0 and c_k for $k \geq 0$ as **fixed**. And we want to ask for which $z \in \mathbb{C}$ does this series converge (absolutely?)
Lemma:
Suppose $\sum_{k=0}^{\infty} c_k(z_1 - z_0)^k$ converges for some $z_1 \neq z_0$.
Then, the sum

$$\sum_{k=0}^{\infty} c_k(z - z_0)^k$$

converges absolutely, $\forall z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$
Proof:
$\sum_{k=0}^{\infty} c_k(z_1 - z_0)^k$ converges, so the terms are **bounded**.
$\exists M \geq 0$ such that $|c_k(z_1 - z_0)^k| \leq M, \forall k$.

$$
\begin{aligned}
|c_k(z-z_0)^k| &= |c_k(z_1-z_0)^k \frac{(z-z_0)^k}{(z_1-z_0)^k} \\
&= M\left|\frac{z-z_0}{z_1-z_0}\right|^k
\end{aligned}
$$

Let $r = |\frac{z-z_0}{z_1-z_0}| < 1$ by hypothesis.

$$|c_k(z-z_0)^k| \le Mr^k$$

$\sum_{k=0}^{\infty} Mr^k$ converges $= M \cdot \frac{1}{1-r}, (r < 1)$
Hence, by Comparison Test,

$$\sum_{k=0}^{\infty} |c_k(z-z_0)^k|$$

converges.

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Let $c_k \in \mathbb{C}, \forall k \ge 0, z_0 \in \mathbb{C}$

$$\sum_{k=0}^{\infty} c_k(z-z_0)^k$$

is called a complex power series centered at z_0.
We are interested in which $z \in \mathbb{C}$ does

$$\sum_{k=0}^{\infty} c_k(z-z_0)^k$$

converge?
Converge absolutely?
The series always converges absolutely at $z = z_0$ (center).
Last time: **Lemma**
Suppose $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges at some $z_1 \ne z_0$, then

$$\sum_{k=0}^{\infty} c_k(z-z_0)^k$$

converges absolutely $\forall z$ such that $|z-z_0| < |z_1-z_0|$.
It follows from the Lemma, if $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ diverges (does not converge) at z_2, then it diverges at all z such that $|z-z_0| > |z_2-z_0|$.
(Because if it did converge at z_3, $|z_3-z_0| > |z_2-z_0|$) Then by the lemma, it would converge absolutely at z_2.

Radius of Convergence
Theorem: Given a complex power series,

$$\sum_{k=0}^{\infty} c_k (z-z_0)^k$$

centered at z_0, there exists $R \in [0, \infty]$ such that if $R = 0$, then the series converges only at $z = z_0$
$R = +\infty$, then the series converges absolutely for all $z \in \mathbb{C}$.
If $0 \underline{<} R < \infty$, then the series converges absolutely on $D(z_0; R)$ and diverges on $\mathbb{C} \setminus \overline{D(z_0; R)}, (|z - z_0| > R)$
And anything can happen on $\partial\left(D(z_0, R)\right) = \{z : |z - z_0| = R\}$ (It might converge absolutely or converge but not absolutely or diverge)
R is called the radius of convergence.
$D(z_0; R)$ is called the **disc** of **convergence**.
Proof:
If the series converges only at z_0, set $R = 0$. Suppose there exists $z_1 \neq z_0$ such that the series converges at z_1. Let $R = \sup\{r > 0 : \text{Series converges on } D(z_0, r)\}$
This is non-empty set of positive real numbers.
Supremum exists as an extended positive real number either $0 < R < \infty$ or $R = +\infty$.
If $R = +\infty$, then the series converges absolutely for all $z \in \mathbb{C}$. Because if $z \in \mathbb{C}$, there exists $w \in \mathbb{C}$, such that $|z - z_0| < |w - z_0|$ and the series converges at w. (Because $R = +\infty$)
So by lemma, the series converges absolutely at z.
Finally, suppose $0 < R < \infty$.
If $|z - z_0| > R$, then the series does not converge because if it did, then by the lemma, it would converge (absolutely) on $D(z_0; \frac{R+|z-z_0|}{2})$, contradicting the definition of R.
If $|z - z_0| < R$, then by the definition of R as a supremum, the series converges on $D(z_0; \frac{|z-z_0|+R}{2})$
Hence, by the lemma, series converges absolutely at z.
Example:

$$\sum_{k=0}^{\infty} (z-z_0)^k, (c_k = 1, \forall k)$$

has radius of convergence $R = 1$

$$\sum_{k=0}^{\infty} w^k$$

converges iff $|w| < 1$.
In this case, we have **divergence** on $\partial\left(D(z_0; 1)\right)$.
Recall: The Ratio Test (Calculus 2)
Let $\sum_{k=0}^{\infty} b_k$ be a series of **positive** real numbers $b_k > 0, \forall k \geq 0$.

Suppose $\lambda = \lim_{k\to\infty} \frac{b_{k+1}}{b_k}$ exists (As a **finite** real number)
Then if $\lambda < 1$, the series converges.
If $\lambda > 1$, the series diverges.
If $\lambda = 1$, the test is inconclusive. (Anything can happen).
We use the ratio test to prove the following theorem.
Theorem:
Let $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ be a complex power series with $c_k \neq 0, \forall k$.
Suppose

$$L = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists in $[0, \infty]$.
Then the power series has radius of convergence $R = \frac{1}{L}$.
i.e If $L = 0$, then $R = +\infty$, if $L = +\infty$, then $R = 0$, if $0 < L < R$, then $0 < R = \frac{1}{L} < \infty$.
Proof: Suppose $z \neq z_0$.
Let $b_k = \left|c_k(z - z_0)^k\right| > 0, \forall k \geq 0$

$$\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{c_{k+1}(z - z_0)^{k+1}}{c_k(z - z_0)^k}\right| = \frac{|c_{k+1}|}{|c_k|}|z - z_0|$$

If $\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = L$
Then $\lambda = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = L \cdot |z - z_0|$
By Ratio Test, series converge if $\lambda = L|z - z_0| < 1 \iff |z - z_0| < \frac{1}{L}$.
Diverges if $\lambda = L|z - z_0| > 1 \iff |z - z_0| > \frac{1}{L}$.
(If $L = 0$, $\lambda = 0$ for any $z \in \mathbb{C} \Rightarrow R = +\infty$)
If $L = +\infty, \lambda = +\infty$, unless $z = z_0$
Ratio Test still works.
Examples:

1. $\sum_{k=0}^{\infty}(-1)^k(z - 1)^k$

$$\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}}{(-1)^k}\right| = 1$$

So $R = \frac{1}{1} = 1$

This series converges absolutely for all z such that $|z-1| < 1$ and diverges if $|z - 1| > 1$ by the theorem.

In this example, we can say more.

$$\sum_{k=0}^{\infty} w^k$$

, $w = -(z - 1), w^k = (-1)^k(z - 1)^k$

Converges absolutely if $|w| = |z - 1| < 1$ and diverges if $|w| \geq 1$.

So in this case we have divergence on boudnary.

If $|w| = |z-1| < 1$, then we know

$$\sum_{k=0}^{\infty} w^k = \frac{1}{1-w}$$

.

$$\sum_{k=0}^{\infty} (-1)^k (z-1)^k = \frac{1}{1-(-(z-1))} = \frac{1}{z}$$

2. $\sum_{k=0}^{\infty} \frac{1}{k!} z^k$, $c_k = \frac{1}{k!}$

$$\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} \right| = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

So by the theorem, $R = +\infty$

So this series converges absolutely for all $z \in \mathbb{C}$.

In fact, we will see next week that this converges to e^z.

Uniform Convergence
This is a type of convergence for sequence of functions.
Let Ω be a domain, let $f_k : \Omega \to \mathbb{C}$ be a sequence of complex-valued functions on Ω.
We say that (f_k) converges pointwise to a function $f : \Omega \to \mathbb{C}$ iff $\forall z \in \Omega, f_k(z) \to f(z)$ as $k \to \infty$.
This means for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, depending on both of ϵ and $z \in \Omega$, such that if $k \geq N$, then $|f_k(z) - f(z)| < \epsilon$.
$N = N(\epsilon, z)$
Example:
$f_k(z) = 1 + z + z^2 + \cdots + z^k, k \geq 0$.

$$\lim_{k\to\infty} f_k(z) = f(z) = \frac{1}{1-z}$$

iff $|z| < 1$.
$N = N(\epsilon)$ definitely depends on z.

25 March 11th

Last time: $S \subseteq \mathbb{C}$ a subset.

$$f_n : S \to \mathbb{C}$$

functions $n \geq 0$ We say f_n converges pointwise to $f : S \to \mathbb{C}$ if $f_n(z) \to f(z), \forall z \in S$.

i.e. given $z \in S$ and $\epsilon > 0$, there exists $N = N(z, \epsilon) \in \mathbb{N}$ such that $n \geq N \Rightarrow |f_n(z) - f(z)| < \epsilon$.

Uniform Convergence

Let $S \subseteq \mathbb{C}$ be a subset.

Let $f_n : S \to \mathbb{C}, n \geq 0$ be a sequence of functions.

We say that (f_n) converges $\underline{\text{uniformly}}$ to a function $f : S \to \mathbb{C}$ iff $\forall \epsilon > 0, \exists N = \underline{N(\epsilon) \in \mathbb{N}}$ such that if $n \geq N$, then

$$|f_n(z) - f(z)|\epsilon, \forall z \in S$$

(Same rate)

i.e. We can choose one N (given ϵ) that works for every point in S.

Remark:

It could happen that (f_n) does not converge uniformly to f on S, but does on some proper subset $S' \subset S$.

Basic idea is that uniform convergence preserves "nice properties".

Remark:

If $(f_n) \to f$ uniformly on S, then $(f_n) \to f$ pointwise.

Theorem:

Let Ω be domain, $f_n : \Omega \to \mathbb{C}$ be a sequence of function $\underline{\text{continuous}}$ on Ω. Suppose (f_n) converges uniformly to $f : \Omega \to \mathbb{C}$, then f is continuous on Ω.

"Uniform limit of continuous functions is continous".

Proof:

Let $z \in \Omega$, we need to show that f is continous at z_0.

Let $\epsilon > 0$,

By uniform convergence, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}, \forall z \in \Omega$$

.

$$|f(z) - f(z_0) = |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)|$$

$$\begin{aligned}
&\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\
&< \frac{\epsilon}{3} + |f_n(z) - f_n(z_0)| + \frac{\epsilon}{3} \text{ if } n \geq N
\end{aligned}$$

Since f_n is continuous, there exists $\delta > 0$ such that if $|z - z_0| < \delta$.

Then $|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$.

So $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

So f continuous at z_0.

We will use this soon to show power series give continuous functions.

First, we consider interchange of uniform limit and complex line integral.

Theorem:

Let Γ be a curve in $\mathbb{C}$. Let $f_n : \Gamma \to \mathbb{C}$ be a sequence of functions continous on Γ.

(So the line integral, $\int_\Gamma f_n(z)dz$ makes sense).
Suppose (f_n) converges uniformly to $f : \Gamma \to \mathbb{C}$ (By the previous theorem, f is continuous on Γ, so $\int_\Gamma f(z)dz$ makes sense.)
Then

$$\int_\Gamma \lim_{n\to\infty} f_n(z)dz = \int_\Gamma f(z)dz = \lim_{n\to\infty} \int_\Gamma f_n(z)dz$$

(Integral of uniform limit = limit of integrals)
Proof:
We need to show that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow$ then

$$|\int_\Gamma f_n(z)dz - \int_\Gamma f(z)dz| = |\int_\Gamma (f_n(z) - f(z))\, dz| < \epsilon$$

Let $L = \text{Length}\,(\Gamma) > 0$.
By uniform convergence, exists $N \in \mathbb{N}$ such that $n \geq N$, then

$$|f_n(z) - f(z)| \leq \frac{\epsilon}{L}, \forall z \in \Gamma$$

By ML inequality,

$$\begin{aligned} |\int_\Gamma (f_n(z) - f(z))\, dz| &\leq \frac{\epsilon}{L} L \\ &= \epsilon \end{aligned}$$

Apply this to power series, let $\sum_{k=0}^{\infty} c_k(z - z_0)^k$ be a complex power series centred at z_0.
Suppose the radius of convergence R is positive, (include $R = +\infty$).
Let $D = D(z_0; R)$, (if $R = +\infty, D = \mathbb{C}$), the series converges absolutely for all $z \in D$.
The sum gives a function $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k, \forall z \in D$
$f : D \to \mathbb{C}$, $f(z)$ is the limit of the partial sums.

$$f_n(z) = \sum_{k=0}^{n} c_k(z - z_0)^k$$

all polynomials hence continous on D, (in fact, continous on $\mathbb{C}$)
Theorem:
For any $0 < \rho < R$, the sequence f_n of partial sums converges uniformly to f on the $\overline{D(z_0; \rho)}$.
Remark:
In general, (f_n) does not converge uniformly to f on $D = D(z_0; R)$. But this theorem says that it always does on $\overline{D(z_0; R)}$ if $0 < \rho < R$.
Before we prove this, we need a lemma:
Lemma: Weierstrass M-test

Let $S \subseteq \mathbb{C}$ be a subset, let $f_n : S \to \mathbb{C}$ be sequence of functions. Let $M_n > 0$ such that

$$|f_n(z)| \leq M, \forall z \in S$$

If $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely and uniformly on S.

Proof:

Let $\epsilon > 0$, there exists N such that $m \geq n \geq N$, then

$$M_n + \cdots + M_m < \epsilon$$

Then,

$$|f_n(z) + \cdots + f_m(z)| \leq |f_n(z)| + \cdots + |f_m(z)|$$

Gives uniform convergence.

$$\leq M_n + \cdots + M_m < \epsilon$$

Gives absolute convergence.

Proof of Theorem:

Let $0 < \rho < R$, let $z_1 \in D$ such that $g < |z_1 - z_0| < R$.

Series converges (absolutely) at z_1, so the terms are bounded.

There exists $M > 0$, such that $|c_k(z_1 - z_0)^k| \leq M, \forall k$

Let $z \in \overline{D(z_0; \rho)}$.

$|z - z_0| \leq \rho$.

$$\begin{aligned} |c_k(z - z_0)^k| &= |c_k(z_1 - z_0)^k|\left|\frac{(z - z_0)^k}{(z_1 - z_0)^k}\right| \\ &\leq M\left|\frac{z - z_0}{z_1 - z_0}\right|^k \leq Mr^k \text{ where } r = \frac{g}{|z_1 - z_0|} < 1 \end{aligned}$$

Since $r < 1$,

$$\sum_{k=0}^{\infty} Mr^k$$

converges.

So by Weierstrass M-test. ($M_n = Mr^n$)

We get that

$$\sum_{k=0}^{\infty} c_k(z - z_0)^k$$

converges absolutely and uniformly for all $z \in \overline{D(z_0; \rho)}$.

Corollary: Let $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ where $z \in D = D(z_0; R)$

Then f is <u>continuous</u> at z.

Proof:

Let $z \in D$, choose $\rho > 0$ such that $|z - z_0| < \rho < R$, $z \in D(z_0; \rho) \subseteq \overline{D(z_0; \rho)}$

By the previous result, $f_n(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ (Continuous) converges <u>uniformly</u> (and absolutely) on $\overline{D(z_0; \rho)}$.

Hence, $f(z) = \lim_{n\to\infty} f_n(z)$ is continuous at z.
On the assignment, we will show an example that $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ does not have to converge uniformly on $D(z_0; R)$.
Next time: We will show power series are **analytic** on $D(z_0; R)$.

26 March 13th

Theorem:
Let $f_n : \Omega \to \mathbb{C}$ be a sequence of **analytic functions** on a domain Ω, suppose (f_n) **converges uniformly** to $f : \Omega \to \mathbb{C}$ on Ω.
Then f is analytic on Ω.
("Uniform limit of analytic functions is analytic")
Proof:
Let $z_0 \in \Omega$. We want to show that f is analytic at z_0.
This is equivalent to showing that f is analytic on an open disc $D = D(z_0; \epsilon) \subseteq \Omega$.
Let Γ be a Jordan curve in D.

$$\int_\Gamma f_n(z)dz = 0, \forall n$$

By the Cauchy Integral Theorem because $\Gamma \cup \mathrm{int}(\Gamma) \subseteq D$.
f_n analytic on D.
Now, (f_n) converges to f uniformly.
Thus,

$$0 = \int_\Gamma f_n(z)dz \to \int_\Gamma f(z)dz = 0$$

$$\int_\Gamma f(z)dz = 0$$

For any Jordan curve Γ in D.
By Morera's Theorem, f is analytic on D.
Corollary:
A complex power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k = f(z)$ defines an **analytic function** on its open disc of convergence $D(z_0; R)$.
Proof:
The partial sums $f_n(z) = \sum_{k=0}^{n} c_k(z-z_0)^k$ are polynomials, hence analyticon $D(z_0; R)$. (in fact on all of $\mathbb{C}$)
Every $z \in D$ is an interior point of a closed subdisc $\overline{D(z_0; \rho)} \subseteq D(z_0; R)$ on which the convergence is uniform.
Remark:
The power series may also converge at some points on the boudnary of the disc of convergence. But analyticity doesn't even make sense since $f(z)$ is not defined on $\mathbb{C} \setminus \overline{D}$
Differentiation of power series

Theorem:
Let $f_n : \Omega \to \mathbb{C}$ be a sequence of analytic functions on Ω. Suppose that (f_n) converges uniformly to $f : \Omega \to \mathbb{C}$ on each **closed subdisc** $\overline{D}$ in Ω. Then the sequence $f_n' : \Omega \to \mathbb{C}$ (converges uniformly) to $f' : \Omega \to \mathbb{C}$ on each closed subdisc $\overline{D}$ in Ω.
Proof:
Let $D = \overline{D(z_0;\epsilon)} \subseteq \Omega, r > 0$.
Let $\Gamma = C(z_0; r+d)$ be a circle of radius $r+d$ centered at z_0 such that

$$\overline{D} \subseteq D(z_0; r+d) \subseteq \Omega$$

Let's see why such a Γ exists.
To see this, let $z \in \partial\overline{D}$.
Since Ω is open, there exists $\epsilon_z > 0$ such that

$$D(z, \epsilon_z) \subseteq \Omega$$

$$\{D(z; \frac{\epsilon_z}{z}); z \in \partial\overline{D}\}$$

This is an open cover of $\partial\overline{D}$, which is compact. So there exists a finite subcover.
That is, there exists $z_1, \ldots, z_N \in \partial\overline{D}$ with $\epsilon_k = \epsilon_{z_k}$ such that

$$\partial\overline{D} \subseteq \bigcup_{k=1}^{N} D(z_k, \frac{1}{2}\epsilon_k)$$

Let $d = \min_{k=1}^{N}\left(\frac{\epsilon_1}{2}, \ldots, \frac{\epsilon_N}{2}\right)$
We claim,

$$D(z_0; r+d) \subseteq \Omega$$

Let $z \in D(z_0, r+d)$ with $|z - z_0| \geq r$.
Let w be the point on $\partial\overline{D}$ where ray from z_0 to z intersects.

$$|z - w| \leq d$$

$w \in \partial\overline{D} \Rightarrow w \in D(z_k, \frac{\epsilon_k}{2})$ for some k.

$$|z - z_k| \leq |z - w| + |w - z_k| < d + \frac{\epsilon}{2} \leq \frac{\epsilon_k}{2} + \frac{\epsilon_k}{2} = \epsilon_k$$

Hence, $D(z_0; r+d) \subseteq \Omega$.
Proves this claim.
Back to the Proof of the Theorem.
Let $\epsilon > 0$, we need to show there exists $N \in \mathbb{N}$ such that $n \geq N$ then

$$|f_n'(\zeta) - f'(\zeta)| < \epsilon, \forall \zeta \in \overline{D}$$

(This says $(f_n') \to f'$ uniformly on $\overline{D}$.
Let $\zeta \in \overline{D}$.

Apply generalized Cauchy Integral Formula to the analytic function $f_n - f$. We get

$$f_n'(\zeta) - f'(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f_n(z) - f(z)}{(z-\zeta)^2}dz$$

For $\zeta \in \overline{D}$ and $z \in \Gamma$, $|z - \zeta| \geq d$.

$$\frac{1}{|z-\zeta|^2} \geq \frac{1}{d^2}, \forall z \in \Gamma, \zeta \in \overline{D}$$

.

$$\left|\frac{f_n(z) - f(z)}{(z-\zeta)^2}\right| \leq \frac{|f_n(z) - f(z)|}{d^2}$$

By uniform convergence of (f_n) to f on $\overline{D}$, there exists N such that $n \geq N$, then

$$|f_n(z) - f(z)| < \frac{\epsilon 2\pi d^2}{\text{Length}(\Gamma)}$$

By ML inequality, if $n \geq N$.

$$\begin{aligned} |f_n'(\zeta) - f'(\zeta)| &\geq \frac{1}{2\pi}\left(\frac{\epsilon 2\pi d^2}{d^2 \text{Length}(\Gamma)}\right)\text{Length}(\Gamma) \\ &= \epsilon \end{aligned}$$

Corollary: Let $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ be a complex power series with positive radius of convergence. Then the complex power series

$$\sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}$$

has the same disc of convergence $D(z_0; R)$. and it converges to $f'(z)$ on $D(z_0; R)$. i.e We can differentiate convergent power series term-by-term inside the open disc of convergence $D(z_0; R)$.

Proof:

By previous result, f is analytic on $D = D(z_0; R)$. Hence, f' exists and is analytic on D.

$$f_n(z) = \sum_{k=0}^{n} c_k(z-z_0)^k$$

$$f_n'(z) = \sum_{k=0}^{n} c_k k(z-z_0)^{k-1}$$

Analytic on D.

By previous theorem, f_n' converges uniformly to f' on any closed subdisc $\overline{D}$ of $D(z_0; R)$.
But

$$\lim_{n\to\infty} f_n'(z) = \sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}$$

The radius of convergence of $\sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}$ is therefore at least R.
We need to show that it's not larger.
Let z_1 satisfy $|z_1 - z_0| > R$. We need to show $\sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}$ does not converge at z_1.
Let $k \in \mathbb{N}$ such that $k > |z_1 - z_0|$, $\frac{k}{|z_1-z_0|} > 1$.

$$|c_k(z-z_0)^k| < |kc_k(z-z_0)^{k-1}|, \forall k \text{ sufficiently large}$$

By Comparison Test, the second series does not converge.
Summary:
Let $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ be a complex power series with positive radius of convergence.

$$D = D(z_0; R)$$

is the open disc of convergence.

- The series converges absolutely and uniformly on any **closed subdisc** of D.

  Let $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ for $z \in D$.

  We showed f is **analytic** on D.

  We also showed that on D, $f'(z) = \sum_{k=0}^{\infty} kc_k(z-z_0)^{k-1}$

  Which has the same open disc of convergence.

  Hence, convergent power series are analytic.

  Next week: We show that analytic functions "are" convergent power series.